

Real Functions

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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1 Introduction

1.1 Motivation

The Riemann integral we have known so far is fairly limited. For example it doesn't allow us to compute the Riemann integral of Dirichlet's function $f: [0, 1] \rightarrow \mathbb{R}$ defined as

$$f(x) = \mathbb{1}_{\mathbb{Q} \cap [0,1]} = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

In his thesis Lebesgue introduced a new type of integral called a Lebesgue integral that allows us to compute integrals for functions like Dirichlet's function, and he continued to develop more concepts like measure, and almost everywhere.

1.2 Motivation for Lebesgue integral

Let $f(x) = \mathbb{1}_A$ be the function that we want $\int \mathbb{1}_A$ to be the volume of the set A .

First we would like we define what is a volume of a set. We would want to require a couple of things

- (1) $\mu(A)$ is defined for all $A \subseteq \mathbb{R}^n$;
- (2) $\mu([0, 1]^n) = 1^n = 1$;
- (3) μ to be invariant to congruations (isometries).
- (4) If $\{A_i\}_{i=1}^{\infty}$ is a countable sequence of pairwise disjoint sets then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Remark 1.1. Property (4) is called σ -additivity.

Theorem 1.1 (Hausdorff, 1914). *There is no function that satisfies (1) – (4) at the same time.*

We will prove this theorem later. For now we can only try to weaken the requirements. For example instead of σ -additivity we might require finite additivity.

Theorem 1.2. *There exists a function that satisfies the wanted requirements in dimensions 1 and 2 but not in dimension $n \geq 3$.*

Remark 1.2. For example in $n = 3$ we have the Banach–Tarski paradox.

Paradox 1.1 (Banach–Tarski, 1924). For every $n \geq 1$ we can divide S^2 in $\mathbb{R}^n$ to a finite amount of parts such that when they are rotated and rearranged, can form a new sphere of any desired size.

Remark 1.3. The bigger sphere we would want to form, the greater is the minimal pieces we need to divide the unit sphere in order to form it.

This paradox is based on the use of the axiom of choice, but since we assume the axiom of choice in this course we still need to modify the requirements.

Instead of (4) we require subadditivity:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

This is possible but instead we would like to keep σ -additivity but instead give up on requirement (0). We will only define volume only for “nice” sets which include all of the sets we work with in analysis or geometry etc. these sets form a σ -algebra on $\mathbb{R}^n$.

2 Algebras and σ -algebras

2.1 Definitions

Definition 2.1 (Algebra). An algebra on a nonempty set X is a collection $\mathcal{A}$ of subsets of X such that

- (1) $\emptyset, X \in \mathcal{A}$;
- (2) $\mathcal{A}$ is closed under complements;
- (3) $\mathcal{A}$ is closed under finite unions and intersections.

Definition 2.2 (σ -algebra). A σ -algebra on a nonempty set X is an algebra $\mathcal{A}$ on X that is also closed under countable unions and intersections.

Remark 2.1. A σ -algebra is sometimes called a σ -field. That is why it is sometimes denoted $\mathcal{F}$.

Remark 2.2. From De-Morgan laws we know that

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

so it is only necessary to require closure under countable unions or countable intersections.

Remark 2.3. It is also possible to require $\mathcal{A}$ to be nonempty instead of (1). Then for $A \in \mathcal{A}$ we have

$$A \in \mathcal{A} \implies A^c \in \mathcal{A} \implies A \cup A^c = X \in \mathcal{A} \text{ and } A \cap A^c = \emptyset \in \mathcal{A}.$$

Example 2.1. $\mathcal{A} = \{\emptyset, X\}$ and $\mathcal{A} = 2^X$ are the smallest and biggest σ -algebras on X respectively.

Example 2.2. If X is not countable. Then

$$\mathcal{A} = \{E \subseteq X : E \text{ or } E^c \text{ are countable}\} \neq 2^X$$

is a σ -algebra.

Definition 2.3 (Cocountability). Let X be a set. Then A is called cocountable if A^c is countable.

Example 2.3 (Generated σ -algebra). Let $F \subseteq 2^X$ be a family of subsets of X . The σ -algebra generated by F is defined as

$$\sigma(F) = \bigcap \left\{ \mathcal{A} \subseteq 2^X : \mathcal{A} \text{ is a } \sigma\text{-algebra and } F \subset \mathcal{A} \right\}.$$

Remark 2.4. Notice that the intersection is indeed a σ -algebra.

Remark 2.5. Let $\mathcal{A}$ be a σ -algebra and $F \subseteq \mathcal{A}$. Then $\sigma(F) \subseteq \mathcal{A}$.

Corollary 2.1. Suppose $F_1 \subseteq \sigma(F_2)$ and $F_2 \subseteq \sigma(F_1)$. Then $\sigma(F_1) = \sigma(F_2)$.

Definition 2.4 (Borel σ -algebra). Let X be a topological space. Then we define the Borel σ -algebra as the σ -algebra generated by the open sets in X .

$$B(X) = \sigma \left(\{G \subset X \mid G \text{ is open}\} \right).$$

Remark 2.6. Recall that G denotes an open set, F a closed set, G_δ a countable intersection of open sets and F_σ a countable union of closed sets. Similarly $G_{\delta\sigma}$ is a countable union of G_δ sets etc.

Remark 2.7.

$$B(X) = \sigma \left(\{F \subset X \mid F \text{ is closed}\} \right).$$

Proposition 2.2. $B(\mathbb{R})$ is generated by the collection of any type of interval.

Proof. Consider the collection of open intervals in $\mathbb{R}$. Since any open set in $\mathbb{R}$ is a countable union of disjoint open intervals, we get that $B(\mathbb{R}) = \sigma((a, b))$.

Consider the collection of closed intervals in $\mathbb{R}$. Since any closed interval is a countable intersection of open sets, we have that $[a, b] \subseteq \sigma((a, b))$. Since any open interval is a countable union of closed sets, we have that $(a, b) \subseteq \sigma([a, b])$.

The proof for other types of intervals (like $(a, b]$) is similar and thus omitted. $\square$

2.2 The product σ -algebra

Definition 2.5 (Product σ -algebra). Let $\{\mathcal{A}_i\}_i$ be a collection of σ -algebras on $\{X_i\}_i$. Then, the product σ -algebra $\bigotimes_{i \in I} \mathcal{A}_i$ is the σ -algebra generated by the cylindrical sets:

$$\mathcal{S} = \left\{ \prod_{i \in I} U_i \mid \exists j \in I \text{ s.t. } U_i = X_i \text{ for } i \in I \setminus \{j\} \text{ and } U_j \in \mathcal{A}_j \right\},$$

which is the also the set that generates the product topology on the product space.

Remark 2.8. In the case of $|I| < \aleph_0$ this is equivalent to the σ -algebra generated by the open sets in $\bigotimes_{i \in I} \mathcal{A}_i$ with respect to the box topology.

Proposition 2.3. $B(\mathbb{R}^n) = \bigotimes_{i=1}^n B(\mathbb{R})$.

Proof. Showing that $\prod_{i=1}^n (a_i, b_i) \subseteq B(\mathbb{R}^n)$ can be done simply by definition. We can also show that $B(\mathbb{R}^n) \subseteq \prod_{i=1}^n (a_i, b_i)$ because for an open set $A \subseteq \mathbb{R}^n$ we can see that it is the countable union of all the open boxes with rational sized edges contained in A around any rational point $q \in A$ (which exist because A is open). $\square$

3 Measure

3.1 Definitions

Definition 3.1 (Measure). Let $\mathcal{A}$ be a σ -algebra on X . A function $\mu: \mathcal{A} \rightarrow [0, \infty]$ is called a measure if

- (1) $\mu(\emptyset) = 0$;
- (2) Given a sequence $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ of disjoint sets we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Remark 3.1. Property (2) is called σ -additivity. It obviously implies finite additivity because we can choose $E_j = \emptyset$.

Definition 3.2 (Measurable space). The space $(X, \mathcal{A})$ is called a measurable space. The elements of $\mathcal{A}$ are called measurable sets.

Definition 3.3 (Measure space). The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Definition 3.4 (Finite measure). A finite measure is a measure μ on X such that $\mu(X) < \infty$.

Definition 3.5 (σ -finite measure). A σ -finite measure is a measure μ on X such that $X = \bigcup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{A}$ such that $\mu(E_i) < \infty$.

Definition 3.6 (Borel measure). A measure μ on a topological space X is called a Borel measure if $B(X) \subseteq \mathcal{A}$.

Example 3.1 (Delta measure). Let $x_0 \in X$ and $\mathcal{A} = 2^X$. Then the delta measure is

$$\delta_{x_0}(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}.$$

Example 3.2 (Counting measure). The counting measure is the measure on $\mathcal{A} = 2^X$ such that $\mu(E) = |E|$.

Example 3.3. Suppose $\aleph_0 < |X|$ and let $\mathcal{A}$ be the σ -algebra of the countable or cocountable subsets of X . Then the following is a measure on X ,

$$\mu(E) = \begin{cases} 0, & |E| \leq \aleph_0 \\ 1, & |E^c| = \aleph_0 \end{cases}.$$

Example 3.4. Suppose $|X| = \infty$ and $\mathcal{A} = 2^X$. Then the following is a measure on X ,

$$\mu(E) = \begin{cases} 0, & |E| < \infty \\ 1, & |E| = \infty \end{cases}.$$

Remark 3.2. The last example is a finite-additive, but not σ -additive.

Proposition 3.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- (1) Let $E, F \in \mathcal{A}$ such that $E \subseteq F$. Then $\mu(E) \leq \mu(F)$.

(2) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

(3) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$ such that $E_1 \subseteq E_2 \subseteq \dots$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$$

(4) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$ such that $E_1 \supseteq E_2 \supseteq \dots$ then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$$

Proof.

(1) We have that $F = E \sqcup (F \setminus E)$ so

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

(2) Define the sets

$$\begin{aligned} F_1 &:= E_1 \\ F_2 &:= E_2 \setminus E_1 \\ &\vdots \\ F_k &:= E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right). \end{aligned}$$

We now have that $\{F_k\}_{k=1}^{\infty}$ are disjoint and for all $n \geq 1$ we have

$$\bigsqcup_{j=1}^n F_j = \bigcup_{j=1}^n E_j$$

which implies that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigsqcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$$

where the last inequality follows from $F_j \subseteq E_j$ and (1).

(3) Define similarly

$$\begin{aligned} F_0 &:= \emptyset \\ F_1 &:= F_1 \setminus E_0 \\ &\vdots \\ F_k &:= F_k \setminus F_{k-1}. \end{aligned}$$

We now have that $\{F_k\}_{k=1}^\infty$ are disjoint and for all $n \geq 1$ we have

$$\bigsqcup_{j=1}^n F_j = \bigcup_{j=1}^n E_j$$

which implies that

$$\mu\left(\bigcup_{j=1}^n E_j\right) = \mu\left(\bigsqcup_{j=1}^n F_j\right) = \sum_{j=1}^\infty \mu(F_j) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(F_j) = \lim_{n \rightarrow \infty} \underbrace{\mu\left(\bigsqcup_{j=1}^n F_j\right)}_{\mu(E_n)}$$

where the last equality follows from additivity.

(4) This can be concluded using De Morgan's laws on (3). □

Definition 3.7 (μ -negligible set). A negligible set is a set $E \in \mathcal{A}$ such that $\mu(E) = 0$.

Definition 3.8 (Almost everywhere). Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that a certain property is true *almost everywhere* if it is true for any $x \in X \setminus E$ such that E is of measure zero (μ -negligible).

Example 3.5. We have that $\sin x \neq 0$ almost everywhere because

$$|E| = \left| \{x \in \mathbb{R} \mid \sin x = 0\} \right| = \aleph_0$$

is μ -negligible for the measure μ from Example 3.3 with $X = \mathbb{R}$ defined

$$\mu(E) = \begin{cases} 0, & |E| \leq \aleph_0 \\ 1, & |E^c| = \aleph_0 \end{cases}.$$

Remark 3.3. A countable union of μ -negligible sets is μ -negligible from subadditivity.

Remark 3.4. A subset of a μ -negligible set is μ -negligible since measures are monotone.

3.2 Complete measure spaces

Definition 3.9 (Complete measure space). A triple $(X, \mathcal{A}, \mu)$ is called a complete measure space if for any $F \in \mathcal{A}$, if $\mu(F) = 0$ then for all $E \subseteq F$ we have $E \in \mathcal{A}$.

Theorem 3.2 (Completion theorem). Let $(X, \mathcal{A}, \mu)$ be a measure space. Define

$$\begin{aligned} \mathcal{N} &:= \{N \in \mathcal{A} \mid \mu(N) = 0\} \\ \overline{\mathcal{A}} &:= \{E \cup F \mid E \in \mathcal{A} \text{ and } F \subseteq N \in \mathcal{N}\}. \end{aligned}$$

Then $\overline{\mathcal{A}}$ is a σ -algebra, and there exists a unique extension $\bar{\mu}$ of μ such that $(X, \overline{\mathcal{A}}, \bar{\mu})$ is a complete measure space.

Proof. In the problems section. □

Remark 3.5. $\overline{\mathcal{A}}$ is the smallest σ -algebra containing $\mathcal{A}$ and all the subsets of μ -negligible sets.

Our goal is to construct a measure space $(\mathbb{R}^n, \mathcal{A}, \mu)$ such that

$$\mu \left(\prod_{i=1}^n [a_i, b_i] \right) = \prod_{i=1}^n |a_i - b_i|.$$

Recall that earlier we saw that

$$B(\mathbb{R}^n) = \bigotimes_{i=1}^n B(\mathbb{R}) = \sigma \left(\prod_{i=1}^n [a_i, b_i] \right)$$

which implies that $B(\mathbb{R}^n) \subseteq \mathcal{A}$. To define the measure for other sets we can use an outer measure

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mu(R_i) \mid E \subseteq \bigcup_{i=1}^{\infty} R_i \right\}.$$

We now have two questions.

- (1) Do we always have $\mu^*(R) = \mu(R)$? Yes.
- (2) Is μ^* a measure? No, it is an outer measure.

Remark 3.6. Defining μ^* only on “good” sets gives us a measure called Lebesgue measure. Good sets are sets for which the outer measure is equal to the inner measure.

Definition 3.10 (Outer measure). Let $X \neq \emptyset$. Then $\mu^*: 2^X \rightarrow [0, \infty]$ is called an outer measure if

- (1) $\mu^*(\emptyset) = 0$;
- (2) μ^* is monotone (if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$);
- (3) μ^* is subadditive $\left(\mu^* \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu^*(E_i) \right)$.

Proposition 3.3. Let $\mathcal{E} \subseteq 2^X$ be a family of subsets of X and $\varphi: \mathcal{E} \rightarrow [0, \infty]$ such that $\emptyset, X \in \mathcal{E}$ and $\varphi(\emptyset) = 0$. For all $A \subseteq X$ we define

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(E_i) \mid E_i \in \mathcal{E} \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

then μ^* is an outer measure.

Remark 3.7. It is not promised that $\mu^*(E) = \varphi(E)$ for $E \in \mathcal{E}$.

Proof.

- (1) We can cover $\emptyset$ by $E_i = \emptyset$.
- (2) Let $A \subseteq B$. We can use the cover of B for A since $A \subseteq B \subseteq \bigcup E_i$ which implies that $\mu^*(A) \leq \mu^*(B)$.
- (3) By the definition of the infimum there exists a covering $A_i \subseteq \bigcup_{k=1}^{\infty} E_k^i$ such that

$$\mu \left(\bigcup_{k=1}^{\infty} E_k^i \right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \cdot 2^{-i}$$

which implies that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{k,i=1}^{\infty} E_k^i$ and so

$$\mu^* \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{k,i} \varphi(E_k^i) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon$$

but $\epsilon > 0$ is arbitrary which completes the proof. □

Remark 3.8. Every outer measure can be constructed by φ in this way.

Definition 3.11 (μ^* -measurable set). A set $A \subseteq X$ is called μ^* -measurable if for all $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark 3.9. It is clear that the $\leq$ direction is always satisfied from subadditivity, so it is only necessary to verify the $\geq$ direction. In particular, it is only necessary to verify it for sets such that $\mu^*(E) < \infty$. Let $A \subseteq E$ be such that $\mu^*(E) < \infty$. Then we need to verify that

$$\underbrace{\mu^*(A)}_{\text{outer measure}} = \underbrace{\mu^*(E) - \mu^*(E \cap A^c)}_{\text{inner measure}}.$$

3.3 Carathéodory's outer measure theorem

Theorem 3.4 (Carathéodory's outer measure theorem). *Let μ^* be an outer measure on X . Then*

- (1) $\mathcal{F}_{\mu^*} := \{A \subseteq X \mid A \text{ is } \mu^*\text{-measurable}\}$ is a σ -algebra.
- (2) The triplet $(X, \mathcal{F}_{\mu^*}, \mu^*|_{\mathcal{F}_{\mu^*}})$ is a complete measure space.

Remark 3.10. We saw in the recitations that in order to show that $\mathcal{F}$ is a σ -algebra, it suffices to prove that if $E_i \in \mathcal{F}$ are disjoint, then

$$\bigsqcup_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Proof.

- (1) Let us show that $\mathcal{F} = \mathcal{F}_{\mu^*}$ is an algebra. $\mathcal{F}$ is closed under the complement operation because by the symmetric definition, if A is μ^* -measurable so is A^c .

Let $A, B \in \mathcal{F}$ and $E \subseteq X$. Since A, B are μ^* -measurable we have that

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \end{aligned}$$

We also have that

$$(A \cup B) \cap E = ((A \cap B) \cap E) \sqcup ((A^c \cap B) \cap E) \sqcup ((A \cap B^c) \cap E).$$

Thus, from subadditivity

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

which implies $A \cup B \in \mathcal{F}$. By induction we get that $\mathcal{F}$ is closed under finite unions which makes it an algebra. Moreover, let $A, B \in \mathcal{F}$ be disjoint sets. Then

$$\mu^*(\underbrace{A \sqcup B}_E) = \mu^*((A \sqcup B) \cap A) + \mu^*((A \sqcup B) \cap A^c) = \mu^*(A) + \mu^*(B)$$

so μ is a finite additive measure on $\mathcal{F}$.

Let $\{A_j\}_{j=1}^\infty \subseteq \mathcal{F}$ be a sequence of disjoint sets. Define

$$B_n := \bigsqcup_{i=1}^n A_i \in \mathcal{F} \quad \text{and} \quad B_\infty := \bigsqcup_{i=1}^\infty A_i.$$

For all $E \subseteq X$ we have

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*((E \sqcup B_n) \cap A_n) + \mu^*((E \sqcup B_n) \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i). \end{aligned}$$

Using this and since μ^* is monotone we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B_\infty^c).$$

Taking $n \rightarrow \infty$ we have

$$\mu^*(E) \geq \sum_{i=1}^\infty \mu^*(E \cap A_i) + \mu^*(E \cap B_\infty^c) \geq \mu^*\left(E \cap \underbrace{\bigcup_{i=1}^\infty A_i}_{B_\infty}\right) + \mu^*(E \cap B_\infty^c) \geq \mu^*(E)$$

which implies that $B_\infty \in \mathcal{F}$ (σ -additive). From Remark 3.10 we have that $\mathcal{F}$ is a σ -algebra.

(2) Let $A \subseteq X$ such that $\mu^*(A) = 0$. Then we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \underbrace{\mu^*(A)}_0 + \mu^*(E) = \mu^*(E)$$

which implies that all the expressions must be equal so

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

which means $A \in \mathcal{F}_{\mu^*}$. Since $\mathcal{F}_{\mu^*}$ contains all μ^* -negligible sets, it is complete which also completes the proof. □

3.3.1 Constructing Lebesgue measure on $\mathbb{R}^n$

Let

$$\mathcal{R} = \left\{ R = \prod_{i=1}^\infty [a_i, b_i] \mid a_i, b_i \in \mathbb{R} \right\},$$

and define

$$\varphi(R) = \text{Vol}(R) = \prod_{i=1}^\infty |a_i - b_i| \quad \text{and} \quad \varphi(\mathbb{R}) = \infty.$$

Let us define the outer Lebesgue measure for $E \subseteq \mathbb{R}^n$

$$m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{Vol}(R_i) \mid E \subseteq \bigcup_{i=1}^{\infty} R_i \text{ s.t. } R_i \in \mathcal{R} \right\}.$$

We saw that it is an outer measure. By Theorem 3.4 the space $(\mathbb{R}^n, \mathcal{F}_{m^*}, m^*|_{\mathcal{L}(\mathbb{R}^n)})$ is a complete measure space. We say that

$$m^*|_{\mathcal{L}(\mathbb{R}^n)} = m \quad \begin{array}{l} \text{is called} \\ \text{Lebesgue measure} \end{array} \quad \text{and} \quad \mathcal{F}_{m^*} = \mathcal{L}(\mathbb{R}^n) \quad \begin{array}{l} \text{is the } \sigma\text{-algebra} \\ \text{of the Lebesgue} \\ \text{measurable sets in } \mathbb{R}^n. \end{array}$$

We need to show that

- (1) For all $R \in \mathcal{R}$ we have $m^*(R) = \text{Vol}(R)$;
- (2) $\mathcal{R} \subseteq \mathcal{L}(\mathbb{R}^n) \iff \mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{R}) \subseteq \mathcal{L}(\mathbb{R}^n)$.

Remark 3.11. This means that every open set (in particular every Borel set) is Lebesgue measurable, and in particular that $m(R) = m^*(R) = \text{Vol}(R)$.

Proof. (1), (2) To be added. □

3.3.2 Construction of Lebesgue–Stieltjes measure

Given a nondecreasing monotonic function $F: \mathbb{R} \rightarrow \mathbb{R}$ which is continuous from the right, we would like to construct a Borel measure μ_F on $\mathbb{R}$, such that $\mu_F((a, b]) = F(b) - F(a)$.

The motivation behind this construction is that given a finite Borel measure μ we can construct $F(x) = \mu((-\infty, x])$ we can easily see that

$$\mu((a, b]) = \mu((-\infty, b] \setminus (-\infty, a]) = F(b) - F(a).$$

The function F is clearly monotonic and continuous from the right. We have shown the direction $\mu \rightarrow F$ which is the motivation for the direction $F \rightarrow \mu$.

First we define an outer measure $\mu_F^*: 2^{\mathbb{R}} \rightarrow [0, \infty]$ where $\mathcal{E} = \{(a, b] \mid a < b\}$ and using the function $\varphi((a, b]) = F(b) - F(a)$ so that

$$\mu_F^*((E)) = \inf \left\{ \sum_{i=1}^{\infty} \varphi((a_i, b_i]) \mid E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\} = \inf \left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \mid E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}.$$

According to Theorem 3.4 the triplet $(\mathbb{R}, \mathcal{L}_F, \mu_F^*|_{\mathcal{L}_F})$ is a complete measure space, where $\mathcal{L}_F$ is the μ_F^* measurable sets. We denote $\mu_F := \mu_F^*|_{\mathcal{L}_F}$. We still need to show that

- (1) for all $a < b$ that $\mu_F((a, b]) = \mu^*((a, b]) = F(b) - F(a)$;
- (2) $(a, b] \in \mathcal{L}_F \iff \mathcal{B}(\mathbb{R}) = \sigma((a, b]) \subseteq \mathcal{L}_F \iff (-\infty, b) \in \mathcal{L}_F$.

3.4 Pre-measure

Definition 3.12 (Pre-measure). Let $\mathcal{A}_0$ be an algebra on X . A function $\mu_0: \mathcal{A}_0 \rightarrow [0, \infty]$ is called a pre-measure if

- (1) $\mu_0(\emptyset) = 0$.

(2) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}_0$ are disjoint and $\bigsqcup_{i=1}^{\infty} E_i \in \mathcal{A}_0$, then

$$\mu_0 \left(\bigsqcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu_0(E_i).$$

Remark 3.12. A pre-measure is not a measure because is not necessarily defined on a σ -algebra.

Remark 3.13. In particular we get that μ_0 is finite-additive and monotone.

Remark 3.14. If μ is a measure on a σ -algebra $\mathcal{A}$, then $\mu|_{\mathcal{A}_0}$ is a pre-measure for every subalgebra $\mathcal{A}_0 \subset \mathcal{A}$.

Theorem 3.5 (Carathéodory's extension theorem). *Let μ_0 be a pre-measure on an algebra $\mathcal{A}_0$, and let $\mathcal{A} = \sigma(\mathcal{A}_0)$. Then there exists a measure μ on $\mathcal{A}$ such that $\mu|_{\mathcal{A}_0} = \mu_0$ defined by $\mu = \mu^*|_{\mathcal{A}}$ where*

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(R_i) \mid E \subseteq \bigcup_{i=1}^n E_i \text{ and } E_i \in \mathcal{A}_0 \right\}$$

is the outer measure induced by μ_0 . Moreover, if μ_0 is σ -finite, that is $X = \bigcup_{j=1}^{\infty} X_j$, $X_j \in \mathcal{A}_0$, and $\mu_0(X_j) < \infty$, then the extension is unique (and the existence and uniqueness are for F_{μ^*}).

Corollary 3.6. *If μ_1 and μ_2 are measures on $(X, \mathcal{A})$, $\mathcal{A} = \sigma(\mathcal{A}_0)$ for some algebra $\mathcal{A}_0 \subset \mathcal{A}$, $\mu_1|_{\mathcal{A}_0} = \mu_2|_{\mathcal{A}_0}$, and μ_1, μ_2 are both σ -finite, then $\sigma_1 = \sigma_2$.*

Proof. To be added □

Corollary 3.7. *Every Borel measure μ on $\mathbb{R}$ that is “locally finite” (for all M we have $\mu([-M, M]) < \infty$) is of the form $\mu = \mu_F$ where μ_F is Stieltjes' measure for some nondecreasing function $F: \mathbb{R} \rightarrow \mathbb{R}$ that is continuous from the right. If μ is finite, then $F(x) = \mu((-\infty, x])$, more generally:*

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases}.$$

Proof. To be added □

Example 3.6.

We can now go back to prove Theorem 3.5.

Proof. □

Remark 3.15. We have that $m(\prod_{i=1}^n (a_i, b_i)) = \prod_{i=1}^n (a_i, b_i)$ because we can approximate each edge with $[a_i + 1/k, b_i - 1/k]$ from inside. We also have for boxes $\{R_i\}_{i=1}^{\infty} \subset \mathcal{R}$

$$m \left(\bigsqcup_{i=1}^{\infty} R_i \right) = \sum_{i=1}^{\infty} \mu(R_i).$$

Remark 3.16. For every $x \in \mathbb{R}^n$ we have $m(\{x\}) = 0$ by definition. This also implies that $m(\mathbb{Q}^n) = 0$ from σ -additivity.

Remark 3.17. Recall again that $m(A) = m^*(A) = 0$ for $A \subseteq \mathbb{R}^n$ if and only if

$$\forall \epsilon \quad \exists \{R_i\}_{i=1}^{\infty} \subseteq \mathcal{R} \text{ s.t. } A \subseteq \bigcup_{i=1}^{\infty} R_i \text{ and } \sum_{i=1}^{\infty} \text{Vol}(R_i) < \epsilon.$$

3.5 The relation between $\mathcal{L}(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$

Definition 3.13 (Regular measure). A Borel measure on $(\mathbb{R}^n, \mathcal{A})$ for any metric or topological space is said to be

- Inner regular if

$$\forall E \in \mathcal{A} \quad \mu(E) = \sup \{ \mu(K) \mid K \subseteq E \text{ and } K \text{ is compact and measurable} \}.$$

- Outer regular if

$$\forall E \in \mathcal{A} \quad \mu(E) = \inf \{ \mu(G) \mid E \subseteq G \text{ and } G \text{ is open and measurable} \}.$$

Proposition 3.8. Given a Borel measure μ on $\mathbb{R}^n$ of the form $\mu = \mu^*|_{\mathcal{A}_{\mu^*}}$ for

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(G_i) \mid E \subseteq \bigcup_{i=1}^{\infty} G_i \text{ and } G_i \in \mathcal{E} \subseteq \{\text{open sets}\} \right\}$$

then μ is inner regular and outer regular.

Remark 3.18. This applies to Lebesgue measure and Lebesgue–Stieltjes measure for $n = 1$.

Corollary 3.9. For a measure space $(\mathbb{R}^n, \mathcal{A}_{\mu^*}, \mu)$ as described in Theorem 3.8 (think about $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), m)$) the following are equivalent:

- (0) $E \in \mathcal{A}_{\mu^*}$
- (1) (a) For all $\epsilon > 0$ there exists an open set $G \supseteq E$ such that $\mu^*(G \setminus E) \leq \epsilon$.
 (b) There exists $V \supseteq E$ such that $V \in G_\delta$ and $\mu(V \setminus E) = 0$.
- (2) (a) For all $\epsilon > 0$ there exists a closed¹ set $F \subseteq E$ such that $\mu^*(E \setminus F) \leq \epsilon$.
 (b) There exists $H \subseteq E$ such that $H \in F_\sigma$ and $\mu(E \setminus H) = 0$.
- (3) (a) For all $\epsilon > 0$ there exist an open G and a closed F such that $F \subseteq E \subseteq G$ such that $\mu(G \setminus F) \leq \epsilon$.
 (b) There exist $F_\delta \ni H \subseteq E \subseteq V \in G_\delta$ such that $\mu(V \setminus H) = 0$.
- (4) For $n = 1$ if $\mu^*(E) < \infty$, then for all $\epsilon > 0$ there exists $U = \bigcup_{i=1}^N (a_i, b_i)$ such that $\mu^*(E \Delta U) \leq \epsilon$.

Proof. To be added □

Corollary 3.10. $\mathcal{L}(\mathbb{R}^n)$ is the minimal completion of $B(\mathbb{R}^n)$ with regards to Lebesgue negligible sets.

Proof. To be added □

Proposition 3.11. Lebesgue measure is invariant under translations. This means that $m^*(A) = m^*(A + h)$ for all $A \subseteq \mathbb{R}^n$ and $h \subseteq \mathbb{R}^n$. It follows that A is Lebesgue measurable if and only if $A + h$ is Lebesgue measurable.

Proof. To be added □

Proposition 3.12. Similarly $m^*(x \cdot A) = |c|^n \cdot m^*(A)$ for all $c \in \mathbb{R}$. It follows that A is Lebesgue measurable if and only if $c \cdot A$ is Lebesgue measurable for $c \neq 0$.

¹If $\mu^*(E) < \infty$ we can choose a compact F .

Proposition 3.13. For every $T \in \text{GL}(n)$ we have that $m^*(TA) = |\det(T)| m^*(A)$.

Proposition 3.14. There exists a set $P \subseteq [0, 1]$ that is not Lebesgue measurable.

Proof. Define an equivalence relation on $\mathbb{R}$

$$x \sim y \iff x - y \in \mathbb{Q}.$$

We see that

$$|\mathbb{R}/\sim| = \left| \{[x_0] = x_0 + \mathbb{Q} \mid x_0 \in \mathbb{R}\} \right| > \aleph_0.$$

Using the *axiom of choice* we can choose a representative in $[0, 1]$ from each equivalence class, and set $P \subseteq [0, 1]$ to be the set of all representatives. By definition we see that

$$P + \mathbb{Q} = \bigsqcup_{p \in P} p + \mathbb{Q} = \bigsqcup_{p \in P} [p] = \mathbb{R}$$

and since $[0, 1]^c = [0, 1] + [-1, 1]^c \supseteq P + \mathbb{Q} \cap [-1, 1]^c$, we see that

$$[0, 1] \subseteq P + (\mathbb{Q} \cap [-1, 1]) \subseteq [0, 1] + [-1, 1] \subseteq [-1, 2].$$

Thus if we set $\{q_i\}_{i=1}^\infty = \mathbb{Q} \cap [-1, 1]$ and define $P_i = q_i + P$ we will get

$$[0, 1] \subseteq \bigsqcup_{i=1}^\infty P_i \subseteq [-1, 2].$$

Since Lebesgue measure is invariant under translations we get that for all $i \geq 1$ that $m(P_i) = m(P)$ and then

$$m([0, 1]) = 1 \leq \sum_{i=1}^\infty m(P_i) = \sum_{i=1}^\infty m(P) \leq 3 = m([-1, 2]).$$

However this is a contradiction since there does not exist a number $m(P) \in [0, \infty]$ such that $1 \leq \sum_{i=1}^\infty m(P) \leq 3$ which completes the proof. $\square$

Remark 3.19. Recall that we defined the Cantor set as

$$C := \left\{ x \in [0, 1] \mid x = \sum_{n=1}^\infty \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0, 2\} \right\}.$$

Proposition 3.15. The Cantor set satisfies the following properties:

- (1) It is compact;
- (2) We have $m(C) = 0$;
- (3) $|C| = \aleph = 2^{\aleph_0}$.

Proof.

- (1) We notice that $C = \bigcap_n K_n$ such that

$$K_n := \bigsqcup \left\{ x_0 + [0, 3^{-n}] \mid x_0 = \sum_{n=1}^\infty \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0, 2\} \right\}.$$

Since K_n is a union of a finite amount of compact sets it is compact, and since C is an intersection of a countable amount of compact sets it is compact.

(2) We have from the monotonicity of m that:

$$0 \leq m(C) \leq m(K_n) = \frac{2^n}{3^n} \xrightarrow{n \rightarrow \infty} 0.$$

(3) Let $x \in C$. Then it has a unique ternary expansion without the digit 1:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0, 2\}$$

so it is clear that

$$|C| = \left| \{0, 2\}^{\mathbb{N}} \right| = 2^{\aleph_0} = \aleph = |[0, 1]|.$$

□

Definition 3.14 (Devil's staircase). We can construct the almost-bijection $f: [0, 1] \rightarrow C$ explicitly. Define $b_n = \frac{a_n}{2} \in \{0, 1\}$ and define

$$y = f(x) := \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

It is clear that $f(C) = [0, 1]$ because every number in $[0, 1]$ has a binary expansion, and f is injective except for the numbers that have two binary expansions. We notice that f is monotonic (nondecreasing) because $x < y$ implies $f(x) \leq f(y)$ and if x, y are points on the boundary of a G_n set we even have $f(x) < f(y)$. We can expand the function such that $f: [0, 1] \rightarrow [0, 1]$ by defining it to be constant on G_n , and this is well defined because $f(x) = f(y)$ for $x, y \in \partial G_n$. Since f is monotonic it can only have a jump discontinuities, but since f is onto $[0, 1]$ it doesn't have such points and thus continuous. The function f is called Cantor's function (or sometimes the Devil's staircase). It is continuous everywhere and has zero derivative almost everywhere, but value still goes from 0 to 1 as its argument reaches from 0 to 1. It is a very important counterexample in analysis.

Proposition 3.16. *There exists a Lebesgue measurable function that is not a Borel set. That is,*

$$\mathcal{L}(\mathbb{R}) \setminus B(\mathbb{R}) \neq \emptyset.$$

Proof. We notice that

$$|B(\mathbb{R})| = \aleph_1 = 2^{\aleph},$$

but we have that

$$|\mathcal{L}(\mathbb{R})| \geq 2^{\aleph_1} > \aleph_1$$

because every subset of C is of measure 0, (and since the measure is complete) it is measurable, and there are $2^{|C|}$ such subsets. □

Proof. Let f be the Cantor function, and define

$$g(y) = \inf \{x \in [0, 1] \mid f(x) = y\}$$

as its inverse function. The function g is strictly increasing, and thus injective, but it is not continuous. □

4 Measurable functions

4.1 Definitions

Definition 4.1 (Inverse image). Given a map $f: X \rightarrow Y$. The inverse image map $f^{-1}: 2^Y \rightarrow 2^X$ is given by

$$f^{-1}(E) = \{x \in X \mid f(x) \in E\}, \quad E \subseteq Y.$$

Remark 4.1. The inverse image preserves taking complements, unions, and intersections of any cardinality.

From this remark we get the following proposition.

Proposition 4.1. Let $\mathcal{A}_Y$ be a σ -algebra on Y . Then $f^{-1}(\mathcal{A}_Y) = \{f^{-1}(E) \mid E \in \mathcal{A}_Y\} \subseteq 2^X$ is a σ -algebra on X .

Definition 4.2 (Measurable function). Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces, and let $f: X \rightarrow Y$ be a function. Then f is called a measurable function if for all $E \in \mathcal{A}_Y$ we have $f^{-1}(E) \in \mathcal{A}_X$. That is $f^{-1}(\mathcal{A}_Y) \subseteq \mathcal{A}_X$.

Remark 4.2. A composition of measurable functions is measurable. This follows from the rule $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Lemma 4.2. Let $\mathcal{A}_Y = \sigma(\mathcal{E})$ such that $\mathcal{E} \subseteq 2^Y$. Then $f: X \rightarrow Y$ is measurable if and only if for all $E \in \mathcal{E}$ we have $f^{-1}(E) \in \mathcal{A}_X$.

Proof. The first direction follows immediately from the definitions. Assume that for all $E \in \mathcal{E}$ we have $f^{-1}(E) \in \mathcal{A}_X$. We notice that the set

$$\left\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{A}_X\right\}$$

is a σ -algebra from Theorem 4.1, and it is given by assumption that it contains $\mathcal{E}$. This implies that it contains $\mathcal{A}_Y = \sigma(\mathcal{E})$ which completes the proof. $\square$

Corollary 4.3. Let X, Y be topological spaces. Then a continuous function $f: (X, B(X)) \rightarrow (Y, B(Y))$ is measurable.

Proof. We have that $B(Y) = \sigma(\{\text{open sets}\})$, but since for every open $G \subseteq Y$ we have that $f^{-1}(G) \subseteq X$ is open, we get that $f^{-1}(G) \in B(X)$ which completes the proof. $\square$

Remark 4.3. Notice that measurable functions are a lot more general than continuous functions because we require $f^{-1}(G)$ to be Borel not necessarily open.

Remark 4.4. If $(X, \mathcal{A})$ is a measurable space and $f: X \rightarrow \mathbb{R}$, we always endow $\mathbb{R}$ with $B(\mathbb{R})$.

Corollary 4.4. The following are equivalent:

- (1) $f: (X, \mathcal{A}) \rightarrow \mathbb{R}$ is measurable.
- (2) For all $a \in \mathbb{R}$ we have $f^{-1}([a, \infty)) \in \mathcal{A}$.
- (3) For all $a \in \mathbb{R}$ we have $f^{-1}((a, \infty)) \in \mathcal{A}$.
- (4) For all $b \in \mathbb{R}$ we have $f^{-1}((-\infty, b]) \in \mathcal{A}$.
- (5) For all $b \in \mathbb{R}$ we have $f^{-1}((-\infty, b)) \in \mathcal{A}$.

Where for example $f^{-1}((-\infty, b)) = \{x \in X \mid f(x) < b\}$.

Definition 4.3 (Borel/Lebesgue measurable function). A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be Borel or Lebesgue measurable (with respect to its domain $\mathbb{R}^n$) if

$$f: \begin{matrix} (\mathbb{R}^n, B(\mathbb{R}^n)) \\ (\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)) \end{matrix} \rightarrow (\mathbb{R}, B(\mathbb{R}))$$

is measurable.

Remark 4.5. We always consider the σ -algebra on the domain of the function, not the range.

Remark 4.6. A continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is Borel (and thus Lebesgue) measurable, and a function g that is equal to f almost everywhere, is also Lebesgue measurable (but not necessarily Borel measurable). That is because if

$$m \left(\underbrace{\{x \in \mathbb{R}^n \mid f(x) \neq g(x)\}}_{N_0} \right) = 0$$

then for any $E \in B(\mathbb{R})$ we have that $g^{-1}(E) = f^{-1}(E) \Delta N$ for some negligible subset N of N_0 .

Example 4.1 (Lebesgue measurable but not Borel measurable). The function $f = 0$ is Borel and Lebesgue measurable, but for E that is Lebesgue measurable and not Borel measurable, such that $m(E) = 0$ the function $g = \mathbb{1}_E$ is not Borel measurable even though it is equal to f almost everywhere.

Remark 4.7. If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ such that f is Borel measurable, g is Borel/Lebesgue measurable, then $f \circ g$ is Borel/Lebesgue measurable (because $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$). However, if f is Lebesgue measurable, then $f \circ g$ might not be measurable at all.

Definition 4.4 (Extended real number line). The extended real number line $\overline{\mathbb{R}}$ is defined as $[-\infty, +\infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$. It can be turned into a totally ordered set by defining $-\infty \leq a \leq +\infty$ for all $a \in \overline{\mathbb{R}}$. Since it is totally ordered, we can endow it with the order topology, which is the topology generated by the subbase of open rays. In this way, the space $\overline{\mathbb{R}}$ is compact, every subset of $\overline{\mathbb{R}}$ has a supremum and an infimum (the infimum of the empty set is $+\infty$, and its supremum is $-\infty$). With this topology $\overline{\mathbb{R}}$ is also homeomorphic to the unit interval $[0, 1]$. Thus, the topology is metrizable, corresponding to the standard metric on the interval. However, there is not metric that is an extension of the standard metric on $\mathbb{R}$.

Remark 4.8. This implies that $B(\overline{\mathbb{R}})$ is defined as the all the sets of the form $B \in B(\mathbb{R})$ or $B \cup \{\infty\}$ or $B \cup \{-\infty\}$ or $B \cup \{-\infty, \infty\}$.

Remark 4.9. A function $f: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}$ is said to be measurable if $f: (X, \mathcal{A}) \rightarrow (\overline{\mathbb{R}}, B(\overline{\mathbb{R}}))$ is measurable.

A function $f: (X, \mathcal{A}) \rightarrow \mathbb{C}$ is said to be measurable if $f: (X, \mathcal{A}) \rightarrow (\mathbb{C}, B(\mathbb{C}))$ is measurable. Since we have that $\mathbb{C} \simeq \mathbb{R}^2$ we have that

$$B(\mathbb{C}) = B(\mathbb{R}^2) = B(\mathbb{R}) \otimes B(\mathbb{R}).$$

This implies that f is measurable if

$$\operatorname{Im}(f): (X, \mathcal{A}) \rightarrow (\mathbb{R}, B(\mathbb{R})) \quad \text{and} \quad \operatorname{Re}(f): (X, \mathcal{A}) \rightarrow (\mathbb{R}, B(\mathbb{R}))$$

are both measurable because

$$(\operatorname{Re} f)^{-1}(A) \cap (\operatorname{Im} f)^{-1}(B) = f^{-1}(A \times B).$$

Proposition 4.5. Let $f, g: (X, \mathcal{A}) \rightarrow \mathbb{R}$ be measurable, and let $c \in \mathbb{R}$. Then functions of the form

- (1) $c \cdot f$,
- (2) $f + g$,
- (3) $f \cdot g$,
- (4) $\frac{f}{g}$ (if $g \neq 0$),

are all measurable as well.

Proof. To be added. □

Remark 4.10. We have that Theorem 4.5 is true for $f, g: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}$ as well when declaring

$$(+\infty) + (+\infty) = +\infty \quad (-\infty) - (-\infty) = -\infty \quad 0 \cdot (\pm\infty) = 0 \quad (+\infty) + (-\infty) = a \in \overline{\mathbb{R}}$$

for some fixed $a \in \overline{\mathbb{R}}$.

Proposition 4.6. Let $f, g: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}$ be measurable. The functions $|f|$, $\max(f, g)$, and $\min(f, g)$ are measurable.

Proof.

(1) This is simply a composition with $x \mapsto |x|$.

(2 + 3) We can see this by noticing that

$$\begin{aligned} \{\max(f, g) < b\} &= \{f < b\} \cap \{g < b\} \\ \{\min(f, g) < b\} &= \{f < b\} \cup \{g < b\}. \end{aligned}$$

or by ??

□

4.2 Pointwise convergence

Proposition 4.7. Let $\{f_i: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}\}_{i=1}^{\infty}$ be a sequence of measurable functions. Then the functions

$$\limsup_{i \rightarrow \infty} f_i \quad \liminf_{i \rightarrow \infty} f_i \quad \sup_{i \rightarrow \infty} f_i \quad \inf_{i \rightarrow \infty} f_i$$

are all measurable.

Remark 4.11. All the limits are pointwise. That is, $(\sup_i f_i)(x) = \sup_i (f_i(x))$.

Proof. To be added (not hard). □

Corollary 4.8. Let $\{f_i: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}\}_{i=1}^{\infty}$ be a sequence of measurable functions that converge pointwise to f . Then $f: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}$ is measurable.

Proof.

$$\lim_{i \rightarrow \infty} f_i = \underbrace{\liminf_{i \rightarrow \infty} f_i}_{\text{measurable}}.$$

□

4.3 Simple functions

Definition 4.5 (Indicator function). An indicator function of a set $E \subseteq X$ is defined as

$$\mathbb{1}_E(x) = \chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & \text{otherwise} \end{cases}.$$

Remark 4.12. $\mathbb{1}_E$ is measurable if and only if E is measurable. That is because

$$\{\mathbb{1}_E \geq b\} = \begin{cases} \emptyset, & b > 1 \\ E, & 0 < b \leq 1 \\ X, & b \leq 0 \end{cases}.$$

Definition 4.6 (Simple function). A function $\phi: (X, \mathcal{A}) \rightarrow \mathbb{R}$ (not $\overline{\mathbb{R}}$) is called simple if

$$\phi(X) = \sum_{j=1}^N c_j \cdot \mathbb{1}_{E_j}, \quad c_j \in \mathbb{R} \text{ and } E_j \in \mathcal{A}.$$

Remark 4.13. We see that ϕ is a simple function if and only if it is measurable and $\text{range}(\phi)$ is a finite set.

Remark 4.14. There could be more than one representation of the same simple function as seen in Definition 4.6. The representation is called standard if $\{c_i\}_{i=1}^\infty$ are all different and E_j are disjoint. This representation is unique because we can set $\{c_i\} = \text{range}(\phi)$ and $E_j = \phi^{-1}(\{c_i\})$.

Remark 4.15. The sum and product of simple functions is simple.

Proposition 4.9. Let $f: (X, \mathcal{A}) \rightarrow [0, \infty]$ be a nonnegative measurable function. Then there exists a sequence of simple functions $\{\phi_n\}$ such that

$$0 \leq \phi_1 \leq \phi_2 \leq \dots$$

is a pointwise increasing function and $\phi_n \nearrow f$ pointwise. Moreover, $\phi_n \xrightarrow{n \rightarrow \infty} f$ uniformly on any set E such that f is bounded on E .

Proof. Given $n \in \mathbb{N}$, we split the interval $(0, 2^n]$ to 2^{2n} subintervals of length 2^{-n} as follows

$$I_{k,n} = (k \cdot 2^{-n}, (k+1) \cdot 2^{-n}] \quad k = 0, 1, 2, \dots, 2^{2n} - 1.$$

and set $J_n = (2^n, \infty)$. We also define

$$F_n = f^{-1}(J_n) \quad \text{and} \quad E_{n,k} = f^{-1}(I_{k,n}) \quad \text{and} \quad \phi_n = \sum_{k=0}^{2^{2n}-1} k \cdot 2^{-n} \cdot \mathbb{1}_{E_{n,k}} + 2^n \cdot \mathbb{1}_{F_n}.$$

The sequence ϕ_n is increasing because we just refine the partitions. We also have that if $f|_E \leq 2^{n_0}$ then for all $n_0 \leq n$ we have that $0 \leq f \cdot \phi_n \leq 2^{-n}$ on E . This shows that $\phi_n \xrightarrow{n \rightarrow \infty} f$ uniformly on any set E such that f is bounded on E . In particular we have a pointwise convergence for any x such that $f(x) < \infty$. If $f(x) = \infty$ then the approximation of ϕ_n is given by $2^n \xrightarrow{n \rightarrow \infty} \infty$ so we have that $\phi_n \nearrow f$ pointwise which completes the proof. $\square$

4.4 Almost everywhere

Recall the following definition.

Definition 4.7 (Almost everywhere). Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that a certain property is true *almost everywhere* if it is true for any $x \in X \setminus E$ such that E is of measure zero (μ -negligible).

Remark 4.16. It is possible that $\overline{B} = \{x \mid \neg P(x)\} \notin \mathcal{A}$. We only require that $\overline{B} \subseteq B \in \mathcal{A}$ such that $\mu(B) = 0$. Alternatively, that $\overline{\mu}(\overline{B}) = 0$ where $(X, \overline{\mathcal{A}}, \overline{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$.

Proposition 4.10. Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let $f, g: X \rightarrow \overline{\mathbb{R}}$. Then if f is measurable, and $f = g$ almost everywhere ($\underbrace{\mu(f \neq g)}_N = 0$), then g is measurable.

Proof. From the completeness of the space $(X, \mathcal{A}, \mu)$ we get that

$$\{g < b\} = [\{f < b\} \cap N^c] \sqcup [\{g < b\} \cap N] \in \mathcal{A}.$$

□

Proposition 4.11. Let $(X, \mathcal{A}, \mu)$ is a complete measure space, let $\{f_n: (X, \mathcal{A}) \rightarrow \overline{\mathbb{R}}\}_{i=1}^{\infty}$ be a sequence of measurable function, and let $f_n \xrightarrow{n \rightarrow \infty} f$ almost everywhere, that is

$$\mu \left(\left\{ x \in X \mid \nexists \lim_{n \rightarrow \infty} f_n(x) \text{ or } \exists \lim_{n \rightarrow \infty} f_n(x) \neq x \right\} \right) = 0.$$

Then f is measurable.

Proof. Define $g = \limsup_{n \rightarrow \infty} f_n$. It is measurable from Theorem 4.7. Since $f = g$ μ -almost everywhere, from completeness f is measurable. □

Proposition 4.12. Let $(X, \overline{\mathcal{A}}, \overline{\mu})$ be the completion of the measure space $(X, \mathcal{A}, \mu)$. Let $\overline{f}: X \rightarrow \overline{\mathbb{R}}$ be $\overline{\mathcal{A}}$ -measurable. Then there exists a $\mathcal{A}$ -measurable function $f: X \rightarrow \mathbb{R}$ such that $\overline{f} = f$ almost everywhere.

Proof. To be added. □

Corollary 4.13. Let $f_n \xrightarrow{n \rightarrow \infty} \overline{f}$ μ -almost everywhere. Then it is possible that $\overline{f}$ is not $\mathcal{A}$ -measurable (if $(X, \mathcal{A}, \mu)$ is not complete) but there exists a measurable function f such that $\overline{f} = f$ almost everywhere, and thus $f_n \xrightarrow{n \rightarrow \infty} f$ μ -almost everywhere.

5 Integration

5.1 Integrals of simple functions

Set a measure space $(X, \mathcal{A}, \mu)$ and denote $\mathcal{L}^+ := \{f: X \rightarrow [0, \infty] \mid f \text{ is measurable}\}$.

Remark 5.1. If $\phi \in \mathcal{L}^+$ is simple, then

$$\phi = \sum_{j=1}^N c_j \cdot \mathbb{1}_{E_j}, \quad c_j \in [0, \infty] \text{ and } E_j \in \mathcal{A}.$$

Definition 5.1 (Integration of simple functions). Let $\phi \in \mathcal{L}^+$ be simple. Then we define

$$\int \phi d\mu = \sum_{j=1}^N c_j \cdot \mu(E_j) \in [0, \infty].$$

Remark 5.2. We can verify that this definition is not dependent on the representation of ϕ . Thus we usually compute this integral using the simple function's standard representation.

Remark 5.3. The convention is that $\underbrace{0 \cdot \infty}_{=c_j \cdot \mu(E_j)} = 0$.

Remark 5.4. We denote

$$\int \phi = \int \phi d\mu = \int \phi(x) d\mu(x) = \int \phi(x) \mu(dx)$$

and for all $A \in \mathcal{A}$

$$\int_A \phi d\mu = \int \underbrace{\phi \cdot \mathbb{1}_A}_{\text{simple}} d\mu.$$

Proposition 5.1. Let $\phi, \psi \in \mathcal{L}^+$ be simple functions. Then

- (1) For all $a \geq 0$ we have $\int a\phi = a \int \phi$.
- (2) $\int(\phi + \psi) = \int \phi + \int \psi$.
- (3) If $\phi \leq \psi$ then $\int \phi \leq \int \psi$.

Proof.

- (1) Immediate result.
- (2)
- (2)

□

Definition 5.2 (Integrability). Let $f \in \mathcal{L}^+$. We define

$$\int f d\mu := \sup \left\{ \int \phi d\mu \mid 0 \leq \phi \leq f \text{ and } \phi \text{ is simple} \right\}.$$

A function $f \in \mathcal{L}^+$ is called integrable with respect to μ if $\int f d\mu < \infty$. We define

$$\int_A f d\mu = \int f \cdot \mathbb{1}_A d\mu.$$

Remark 5.5. This definition generalized the previous definition for $f = \phi$. This follows from property (3) in the previous proposition.

Remark 5.6. One may ask why don't we approximate the integral from above? The answer is that while for bounded functions with $|A| < \infty$ we can equivalently define

$$\int f d\mu := \inf \left\{ \int \phi d\mu \mid 0 \leq f \leq \phi \text{ and } \phi \text{ is simple} \right\}.$$

However, the situation is not so simple if the function is unbounded or $|A| = \infty$. Here are a few examples:

- The function $f(x) = x^{-2}$ is integrable over $A = (1, \infty)$. However, for any simple function such that $f \leq \phi$ we have $\int_A \phi d\mu = \infty$ which means it is not integrable.
- The function $f(x) = x^{-0.5}$ is integrable over $A = (0, 1)$. However, there's no simple function s such that $f \leq s$ and s is finite almost everywhere.

Proposition 5.2. Let $\psi, \phi, \phi_1, \phi_2 \in \mathcal{L}^+$. Then

- (1) For all $a \geq 0$ we have $\int a\phi = a \int \phi$.
- (2) $\int(\phi_1 + \phi_2) = \int \phi_1 + \int \phi_2$.
- (3) If $0 \leq \psi \leq \phi$ then $0 \leq \int \psi \leq \int \phi$.

5.2 Monotone convergence theorem

Theorem 5.3 (Monotone convergence theorem). Let $\{f_n\} \subseteq \mathcal{L}^+$ be a nondecreasing sequence of functions that converges pointwise to f . Then

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Remark 5.7. The function f is measurable because $f = \sup_n f_n$.

Remark 5.8. The theorem is false for the Riemann integral. Since $\mathbb{Q}$ is countable we can construct a well order on it such that $\{q_k\}_{k=1}^\infty = \mathbb{Q}$ and then define

$$f_n(x) = \begin{cases} 1, & x \in \bigcup_{k=1}^n q_k \\ 0, & \text{otherwise} \end{cases}$$

but the limit function is $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ which is not Riemann integrable.

Proof. From property (3) of Theorem 5.2 we have

$$\int f_1 \leq \int f_2 \leq \cdots \leq \int f_n \leq \int f$$

so the sequence $\int f_n$ converges and we have

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu.$$

Now set $\alpha \in (0, 1)$ and let ϕ be a simple function such that $0 \leq \phi \leq f$. We define

$$A_n = \{x \in X : f_n(x) \geq \alpha \cdot \phi(x)\}$$

and see that they are measurable, increasing, and $\bigcup A_n = X$. Thus

$$\int f_n \geq \int f_n \mathbb{1}_A \geq \int \alpha \phi \cdot \mathbb{1}_{A_n} = \alpha \int_{A_n} \phi.$$

and then

$$\int_{A_n} \phi = \int \phi \mathbb{1}_{A_n} = \sum_{j=1}^N c_j \mu(E_j \cap A_n) = \sum_{j=1}^N c_j \mu(E_j) = \int \phi.$$

Thus finally we have

$$\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi$$

for an arbitrary $\alpha \in (0, 1)$ so we can take $\alpha \rightarrow 1$ and get

$$\lim_{n \rightarrow \infty} \int f_n \geq \int \phi.$$

We can now take $\sup_\phi$ and get that

$$\lim_{n \rightarrow \infty} \int f_n \geq \int f$$

which completes the proof. $\square$

Corollary 5.4. *An equivalent definition to the integral of $f \in \mathcal{L}^+$ is*

$$\int f d\mu = \lim_{n \rightarrow \infty} \int \phi_n d\mu$$

for any sequence of positive simple functions $0 \leq \phi_n \rightarrow f$. This is because we have shown that there exists such a sequence and the limit is not dependent on the choice of sequence.

Corollary 5.5. *For $f, g \in \mathcal{L}^+$ we have that*

$$\int f + g = \int f + \int g.$$

Proof. There exists sequences of simple functions $\phi_n \nearrow f$ and $\psi_n \nearrow g$, and thus $\phi_n + \psi_n \nearrow f + g$ and then from MCT we have

$$\int f + g = \lim_{n \rightarrow \infty} \left(\int \phi_n + \int \psi_n \right) = \int f + \int g.$$

$\square$

Corollary 5.6 (Switching integrals and sums). *For $\{f_n\}_{n=1}^\infty \subseteq \mathcal{L}^+$ we have*

$$\int \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int f_n.$$

Proof. We can show this is true for a finite n by induction on g , and in the general case we have

$$\int \sum_{n=1}^\infty f_n = \int \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n \stackrel{\text{MCT}}{=} \lim_{N \rightarrow \infty} \int \sum_{n=1}^N f_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int f_n = \sum_{n=1}^\infty \int f_n$$

$\square$

Remark 5.9. Recall that in general, if $f_n \xrightarrow{\text{pointwise}} f$ we don't necessarily have that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

The limit might not even exist. For example, we can choose $f_n = \mathbb{1}_{[n, n+1)}$ with $\mu = m$ being Lebesgue measure. We have that $f_n \xrightarrow{n \rightarrow \infty} 0$ pointwise, but for all $n \geq 1$ we have $\int f_n dm = 1$ and thus

$$1 = \lim_{n \rightarrow \infty} \int f_n dm \neq \int 0 dm = 0.$$

Remark 5.10. The following lemma shows us that even if the monotonic convergence theorem does not hold for non monotonic function sequences, at least one part of it always holds.

Lemma 5.7 (Fatou's lemma). *Let $\{f_n\}_{n=1}^\infty \subseteq \mathcal{L}^+$. Then*

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Remark 5.11. The direction of the inequality is affected by $\{f_n\}_{n=1}^\infty \subseteq \mathcal{L}^+$. For nonpositive functions we have

$$\int \limsup_{n \rightarrow \infty} f_n \geq \limsup_{n \rightarrow \infty} \int f_n.$$

Proof. Set k , and then for all $j \geq k$ we have that that

$$\inf_{n \geq k} f_n \leq f_j \implies \int \inf_{n \geq k} f_n \leq \int f_j \implies \int \inf_{n \geq k} f_n \leq \inf_{j \geq k} \int f_j \implies$$

TO BE CONTINUED □

Proposition 5.8. *Let $f \in \mathcal{L}^+$. Then $\int f d\mu = 0$ if and only if $f = 0$ μ -almost everywhere.*

Proof. First suppose that ϕ is simple.

$$\phi = \sum_{j=1}^N c_j \cdot \mathbb{1}_{E_j}.$$

This case is clear because

$$\begin{aligned} 0 = \int \phi d\mu &= \sum_{j=1}^N c_j \cdot \mu(E_j) \iff \forall j \quad c_j \mu(E_j) = 0 \\ &\iff c_j = 0 \text{ or } \mu(E_j) = 0 \iff \phi = 0 \text{ } \mu\text{-almost everywhere.} \end{aligned}$$

Now for a general function, first assume that $f = 0$ almost everywhere. Then if $0 \leq \phi \leq f$ is simple then $\phi = 0$ almost everywhere and thus $\int \phi d\mu = 0$ so $\int f = \sup \int \phi d\mu = 0$.

For the second direction we need to use Markov's inequality.

Lemma 5.9 (Markov's inequality). *Let $f \in \mathcal{L}^+$. Then for all $r > 0$ we have*

$$\mu(f \geq r) \leq \frac{\int f d\mu}{r}.$$

Proof.

$$r \cdot \mu(f \geq r) = \int r \cdot \mathbb{1}_{\{f \geq r\}} d\mu \leq \int f d\mu.$$

□

Define $A_n = \left\{f \geq \frac{1}{n}\right\}$. Assume by contradiction that we don't have $f = 0$ almost everywhere, then there would exist n such that $\mu(A_n) > 0$, because otherwise we would have

$$\mu(f > 0) = \mu\left(\bigcup A_n\right)$$

and from continuity from below this is equal to $\lim_{n \rightarrow \infty} \mu(A_n) = 0$ so $f = 0$ everywhere which is a contradiction. Thus,

$$f \geq \frac{1}{n} \cdot \mathbb{1}_{A_n} \implies \int f \geq \frac{1}{n} \cdot \mu(A_n) > 0$$

in contradiction, which completes the proof. $\square$

Corollary 5.10. *In the convergence theorems (MCT/Fatou) we can require convergence in μ -almost everywhere instead of pointwise convergence.*

Proposition 5.11. *Let $f \in \mathcal{L}^+$ such that $\int f d\mu < \infty$. Then f is finite μ -almost everywhere ($\mu(\{f = \infty\}) = 0$).*

Proof. The set $E = \{f = +\infty\}$ is measurable, and $f \geq t \cdot \mathbb{1}_E$ for all t so

$$\int f \geq t \int \mathbb{1}_E = t \cdot \mu(E).$$

So if $\mu(E) > 0$ we can take $t \rightarrow \infty$ and get a contradiction. $\square$

Exercise 5.1. If $f \in \mathcal{L}^+$ and $\int f d\mu < \infty$ then $\{f > 0\}$ is σ -finite.

5.3 Integrals of general measurable functions

For each $f: X \rightarrow [-\infty, \infty]$ we define

$$f^+, f^-: X \rightarrow [0, \infty]$$

by

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = \max\{-f, 0\}.$$

It is clear that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. We also have that f is measurable if and only if f^- and f^+ are measurable. They are also nonnegative so if f is measurable this means they are in $\mathcal{L}^+$ so we can also take their integrals.

Definition 5.3 (Integrability). Let $f: X \rightarrow [-\infty, \infty]$ be measurable, and define

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

if at least one of the integrals is finite. If both integrals are finite, we say that f is integrable (or Lebesgue integrable if $\mu = m$ is Lebesgue measure). In the case both integrals are finite, we have

$$\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu < \infty.$$

That is, unlike the Riemann integral, f is integrable if and only if f is absolutely integrable!

Example 5.1 (Integral exists but the function is not integrable). Define $f(x) = \frac{1}{x} \cdot \mathbb{1}_{[0,1]}$,

$$\int_0^1 f dm \stackrel{\text{MCT}}{=} \lim_{\epsilon_n \searrow 0} \int_{\epsilon_n}^1 \frac{1}{x} dm = \lim_{\epsilon_n \searrow 0} \ln x \Big|_{\epsilon_n}^1 = \infty.$$

Thus, the integral exists, but the function is not integrable.

Definition 5.4 ($\tilde{\mathcal{L}}^1$ -space).

$$\tilde{\mathcal{L}}^1(\mu) := \left\{ f: X \rightarrow [-\infty, \infty] \mid \int |f| d\mu < \infty \text{ and } f \text{ is measurable} \right\}.$$

Proposition 5.12. *The space $\tilde{\mathcal{L}}^1$ is a linear space over $\mathbb{R}$, and the integral is a linear functional on it. Let $f, g \in \tilde{\mathcal{L}}^1(\mu)$. Then*

- (1) $\int cf d\mu = c \int f d\mu$.
- (2) $\int f + g = \int f + \int g$.
- (3) $f \leq g \implies \int f \leq \int g$.
- (4) $|\int f| \leq \int |f|$.

Proof. $\tilde{\mathcal{L}}^1(\mu)$ is a linear space because for $f, g \in \tilde{\mathcal{L}}^1(\mu)$ we have

$$\int |af + bg| \leq \int |a||f| + \int |b||g| \implies af + bg \in \tilde{\mathcal{L}}^1(\mu)$$

We can now prove the properties as mentioned in the proposition

- (1) It is clear that

$$\int c \cdot f = \int (cf)^+ - \int (cf)^- = c \left(\int f^+ - \int f^- \right) = c \int f.$$

- (2) Denote $h = f + g$. We have that

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

which implies that

$$\begin{aligned} h^+ + f^- + g^- = f^+ + g^+ + h^- &\implies \int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^- \\ &\implies \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- \end{aligned}$$

- (3) Suppose that $g - f \geq 0$, then

$$0 \leq \int (f - g) = \int f - \int g$$

- (4) We have that $-|f| \leq f \leq |f|$ so according to (3) we have that

$$-\int |f| \leq \int f \leq \int |f|$$

□

Remark 5.12. Let $f: X \rightarrow \mathbb{C}$. Then f is integrable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable because

$$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

Proposition 5.13. *Let $f, g \in \tilde{\mathcal{L}}^1$. Then*

- (1) $\{f \neq 0\}$ is σ -finite.

- (2) $f = g$ μ -almost everywhere if and only if $\int (f - g)d\mu = 0$ if and only if for all $E \in \mathcal{A}$ we have $\int_E f d\mu = \int_E g d\mu$.

Proof.

- (1) We have that $\{f \neq 0\} = \{|f| \neq 0\}$. Since $|f| \in \mathcal{L}^+$ we have $\int |f| d\mu < \infty$.
- (2) $f - g = 0$ μ -almost everywhere if and only if $|f - g| = 0$ μ -almost everywhere if and only if $\int |f - g| d\mu = 0$.

□

Proposition 5.14. Let $f, g \in \tilde{\mathcal{L}}^1(\mu)$. Then $f = g$ μ -a.e. if and only if $\int |f - g| d\mu = 0$ if and only if $\int_E f = \int_E g$ for all $E \in \mathcal{A}$.

Proof. To be added.

□

Definition 5.5 ($\mathcal{L}^1(\mu)$ space). We define an equivalence relation on $\tilde{\mathcal{L}}^1(\mu)$ by

$$f \sim g \iff f = g \text{ } (\mu \text{ almost everywhere}).$$

The transitivity follows from the fact that $\mu(A \cup B) \leq \mu(A) + \mu(B) = 0$. We define

$$\mathcal{L}^1(\mu) = \tilde{\mathcal{L}}^1(\mu) / \sim.$$

Remark 5.13. The space $\mathcal{L}^1(\mu)$ is a vector space on $\mathbb{R}$.

Definition 5.6 ($\mathcal{L}^1(\mu)$ norm). Let $f \in \mathcal{L}^1(\mu)$ be a representative of an equivalence class $[f]$. We define

$$\|[f]\|_{\mathcal{L}^1(\mu)} = \|f\|_{\mathcal{L}^1(\mu)} = \int |f| d\mu (< \infty).$$

Remark 5.14. The norm is well defined by Theorem 5.14.

Remark 5.15. The space $(\mathcal{L}^1(\mu), \|\cdot\|_{\mathcal{L}^1(\mu)})$ is a complete normed space.

Remark 5.16. By Theorem 4.12 there exists a bijection between $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\bar{\mu})$. This means that for all $\bar{f}$ there exists f such that $f \sim \bar{f}$ and thus we identify them as the same space.

5.4 Dominant convergence theorem

Theorem 5.15 (Dominant convergence theorem). Let $\{f_n\} \subseteq \mathcal{L}^1(\mu)$ be a sequence such that

- $f_n \rightarrow f$ μ -almost everywhere.
- There exists an integrable “dominant” $g \in \mathcal{L}^1(\mu)$ such that $|f_n| \leq g$ μ -almost everywhere for all n .

Then $f \in \mathcal{L}^1(\mu)$ and

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Remark 5.17. The second requirement is the analog of the finiteness requirement in the continuity from above property of measures.

Proof. To be added.

□

Proposition 5.16. *Let $\{f_k\} \subseteq \mathcal{L}^1(\mu)$ be a sequence such that*

$$\sum_{k=1}^{\infty} \int |f_k| d\mu < \infty.$$

Then

- (1) $\sum_{k=1}^{\infty} |f_k| \in \mathcal{L}^1(\mu)$.
- (2) *There exists $g \in \mathcal{L}^1(\mu)$ such that $\sum_{k=1}^n f_k \xrightarrow{n \rightarrow \infty} g$ μ -almost everywhere.*
- (3) $\int g d\mu = \sum_{k=1}^{\infty} \int f_k d\mu$.

Proof.

(1)

(2)

(3)

□

6 Covergence

6.1 Types of convergence

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(0) $f_n \xrightarrow{n \rightarrow \infty} f$ **uniformly** if

$$\sup_{x \in X} |f_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0.$$

(1) $f_n \xrightarrow{n \rightarrow \infty} f$ **μ -almost everywhere** if

$$\mu \left(\left\{ x \in X \mid f_n(x) \not\xrightarrow{n \rightarrow \infty} f(x) \right\} \right) = 0.$$

(2) $f_n \xrightarrow{n \rightarrow \infty} f$ **in $\mathcal{L}^1(\mu)$** if

$$\|f_n - f\|_{\mathcal{L}^1(\mu)} = \int |f_n - f| d\mu \xrightarrow{n \rightarrow \infty} 0.$$

(3) $f_n \xrightarrow{\mu} f$ **in measure** if

$$\forall \epsilon > 0 \quad \mu \left(\left\{ x \in X : |f_n(x) - f(x)| \geq \epsilon \right\} \right) \xrightarrow{n \rightarrow \infty} 0.$$

Remark 6.1. We have that (2) $\Rightarrow$ (3) by Markov's inequality.

Remark 6.2. We have that (3) $\Rightarrow$ (1) for some subsequence. There exists a subsequence $\{n_j\}$ such that $f_{n_j} \xrightarrow{n \rightarrow \infty} f$ μ -almost everywhere.

Remark 6.3. We have that (1) $\nRightarrow$ (2) and (2) $\nRightarrow$ (1).

Example 6.1. There are examples here.

6.2 Egoroff's theorem

Remark 6.4. The following theorem tells us when (1) $\Rightarrow$ (0).

Theorem 6.1 (Egoroff's theorem). *Let $\mu(X) < \infty$ and $f_n \xrightarrow{n \rightarrow \infty} f$ μ -almost everywhere. Then for all $\epsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on E^c .*

Proof. To be added □

7 Approximations of functions in $\mathcal{L}^1(\mu)$

Definition 7.1 (Support). Let $f: X \rightarrow \mathbb{R}$ be a function. The set-theoretic support of f is

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

The (topological) support of f is

$$\text{supp}(f) = \overline{f^{-1}(\{0\}^c)}.$$

Proposition 7.1. *The set of simple functions is dense in $\mathcal{L}^1(\mu)$. That is, we can choose $\phi = \sum_{j=1}^k c_j \mathbb{1}_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. We have $\phi \in \mathcal{L}^1(\mu)$ because $\|\phi\| \leq \underbrace{\|f\|}_{\text{finite}} + \underbrace{\|\phi - f\|}_{\epsilon} < \infty$.*

Proposition 7.2. *Let μ be the Lebesgue–Stieltjes measure on $\mathbb{R}$. Then*

- (1) *We can choose E_j to be a finite disjoint union of open intervals for ϕ .*
- (2) *The set of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with compact support (also denoted $C_c(\mathbb{R})$) is dense in $\mathcal{L}^1(\mu)$.*

Proposition 7.3. *Let $\mu = m$ in $\mathbb{R}^n$ (or any other Borel regular measure). Then*

- (1) *We can choose E_j to be a disjoint union of open boxes for ϕ .*
- (2) *The set $C_c(\mathbb{R}^n)$ (continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support) is dense in $\mathcal{L}^1(m)$.*

Proof. To be added □

Remark 7.1. The following theorem shows us that a measurable function is “almost continuous”.

Theorem 7.4 (Lusin’s theorem). *Let $f: [a, b] \rightarrow \mathbb{R}$ be Lebesgue measurable. Then for all $\epsilon > 0$ there exists a compact $E \subseteq [a, b]$ such that $m([a, b] \setminus E) \leq \epsilon$, and $f|_E$ is continuous.*

Remark 7.2. Being continuous on a subset of the domain $E \subset [a, b]$ means that for any $\{x_n\}_{n=1}^\infty \subset E$ such that $x_n \rightarrow x'$ then $\lim_{n \rightarrow \infty} f(x_n) = f(x')$.

Proof. In the problems section. □

Theorem 7.5. *Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.*

- (1) *If f is Riemann integrable, then f is Lebesgue measurable (and thus also Lebesgue integrable), and $\int_a^b f(x) dx = \int_{[a,b]} f dm$.*
- (2) *f is Riemann integrable if and only if $m(\{x \in [a, b] \mid f \text{ has a discontinuity at } x\}) = 0$.*

Proof. To be added. □

Remark 7.3. This theorem is not true on a noncompact interval.

Example 7.1. Define the function

$$f(x) := \sum_{n=1}^{\lfloor x \rfloor} \frac{(-1)^n}{n} \cdot \mathbb{1}_{[n, n+1)}.$$

Since f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable and the harmonic series diverges, we get that $f \notin \mathcal{L}^1(m)$. However, we have that

$$\begin{aligned} \int_1^\infty f(x)dx &= \lim_{R \rightarrow \infty} \int_1^R f(x)dx \\ &= \lim_{R \rightarrow \infty} \sum_{n=1}^{\lfloor R/2 \rfloor} \left(\frac{1}{2n} - \frac{1}{2n+1} \right) \sim \sum_{n=1}^\infty \frac{1}{n^2} < \infty. \end{aligned}$$

Remark 7.4. This theorem is not true on an unbounded function.

Example 7.2. Define the function

$$f(x) := \sum_{n=1}^{\lfloor x \rfloor} (-1)^n \cdot n \cdot \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}.$$

We have that

$$\int_0^1 f(x)dx = \lim_{\epsilon \rightarrow 0} \int_\epsilon^1 f(x)dx < \infty$$

where the last inequality follows from conditional convergence. However, $|f|$ is not Lebesgue integrable since the integral diverges for it.

8 Product measures

Definition 8.1 (Product σ -algebra). Let $(X, \mathcal{A}_X, \mu_X)$ and $(Y, \mathcal{A}_Y, \mu_Y)$ be measure spaces. We define their product measure $\mathcal{A}_X \otimes \mathcal{A}_Y$ to be

$$\mathcal{A}_X \otimes \mathcal{A}_Y := \sigma(A \times B \mid A \in \mathcal{A}_X \text{ and } B \in \mathcal{A}_Y).$$

We would like to construct a product measure $\mu_X \otimes \mu_Y$ on the product measurable space $(X \times Y, \mathcal{A}_X \otimes \mathcal{A}_Y)$ such that

$$\mu_X \otimes \mu_Y(A \times B) = \mu_X(A) \cdot \mu_Y(B).$$

For example, if we denote Lebesgue measure on $\mathbb{R}^n$ as m^n , we would like to have $m^1 \otimes m^1 = m^2$ and $\bigotimes_{i=1}^n m^1 = m^n$ etc.

Exercise 8.1. Show that $m^2(A \times B) = m^1(A) \cdot m^1(B)$.

Before we do that, we need a couple more definitions.

Definition 8.2 (Semialgebra). A set $S \subseteq P(X)$ is a semialgebra over X if

- (1) $\emptyset \in S$;
- (2) $E, F \in \mathcal{E} \implies E \cap F \in \mathcal{E}$;
- (3) $E \in \mathcal{E} \implies E^c$ is a finite disjoint union of elements in $\mathcal{E}$.

Definition 8.3 (Measurable box). A measurable box is a set of the form $A \times B \subseteq X \times Y$ such that $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. We denote this family with $\mathcal{E}$.

Remark 8.1. The set of measurable boxes $\mathcal{E} \subseteq P(X \times Y)$ is a semialgebra over $X \times Y$.

We saw in the recitations that a finite disjoint union of elements of a semialgebra forms an algebra, thus a finite disjoint union of measurable boxes forms an algebra. For this section we now set $\mathcal{A} = \sigma(\mathcal{E})$. To be added clarification

Definition 8.4 (The π function). We define the function $\pi: \mathcal{E} \rightarrow [0, \infty]$ with

$$\pi(A \times B) := \mu_X(A) \cdot \mu_Y(B).$$

Proposition 8.1. Let $A \times B = \bigsqcup_{i=1}^{\infty} A_i \times B_i$. Then $\pi(A \times B) = \sum_{i=1}^{\infty} \pi(A_i \times B_i)$.

Proof. First we notice that

$$\mathbb{1}_A(x) \cdot \mathbb{1}_B(y) = \mathbb{1}_{A \times B}(x, y) = \sum_{i=1}^{\infty} \mathbb{1}_{A_i \times B_i}(x, y) = \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(x) \cdot \mathbb{1}_{B_i}(y).$$

If we consider y as a parameter and integrate by $d\mu_X$ we get

$$\begin{aligned} \mu_X(A) \cdot \underbrace{\mathbb{1}_B(y)}_{\text{constant}} &= \int \mathbb{1}_A(x) \cdot \mathbb{1}_B(y) d\mu_X \stackrel{\text{MCT}}{=} \sum_{i=1}^{\infty} \int \mathbb{1}_{A_i}(x) \cdot \mathbb{1}_{B_i}(y) d\mu_X \\ &= \sum_{i=1}^{\infty} \mu_X(A_i) \cdot \mathbb{1}_{B_i}(y). \end{aligned}$$

We can now integrate by $d\mu_Y$ and get

$$\mu_X(A) \cdot \mu_Y(B) = \sum_{i=1}^{\infty} \mu_X(A_i) \cdot \mu_Y(B_i)$$

which completes the proof. □

Remark 8.2. This implies that π is σ -additive and in particular additive on $\mathcal{E}$.

Definition 8.5 (The extended π function). We define the function $\pi: \mathcal{A} \rightarrow [0, \infty]$ with

$$\pi \left(\bigsqcup_{i=1}^N A_i \times B_i \right) := \sum_{i=1}^N \pi(A_i \times B_i).$$

Proposition 8.2. *This is a good definition.*

Proof. To be added. □

Corollary 8.3. *The function $\pi: \mathcal{A} \rightarrow [0, \infty]$ is a pre-measure.*

Proof. To be added. □

According to Carathéodory's extension theorem (Theorem 3.5), π induces an outer measure $(\mu_X \otimes \mu_Y)^*$ on $X \times Y$, so that its restriction to $\sigma(\mathcal{A}) = \mathcal{A}_X \otimes \mathcal{A}_Y$, which we will denote with $\mu_X \otimes \mu_Y$ is a measure that extends π .

More preciesly we have that for all $E \subseteq X \times Y$ that

$$(\mu_X \otimes \mu_Y)^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mu_X(A_i) \cdot \mu_Y(B_i) \left| E \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i \text{ s.t. } A_i \in \mathcal{A}_X \text{ and } B_i \in \mathcal{A}_Y \right. \right\}.$$

Definition 8.6 (Bounded variation).