

Miscellaneous Projects

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Based on lectures by

Spring 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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0 Introductory note

This document is a simply a conglomerate of notes I did not finish, and probably never will. They are completely unrevised, and thus I do not encourage learning from them. If you try to read them anyway, good luck.

1 Number Theory

1.1 PROPs

Remark 1.1. All rings and rigs are commutative with unit.

Definition 1.1 (PROP). A PROP is a symmetric monoidal category generated by a single object.

Example 1.1. Let R be a ring (or a rig). We can define the prop generated by R as follows:

- (1) the objects of the category are $\bigoplus_{i=1}^n R = R^n$.
- (2) the arrows of the category are $\text{Hom}(R^m, R^n) = \text{Mat}_{n,m}(\mathbb{R})$.

Morphisms of a certain prop are also called props, just like elements of **Grp** are groups.

Example 1.2 (Field with one element). The prop of the field with one element is the category with objects $\{X^n\}_{n \in \mathbb{N}}$ for some arbitrary X , so that $\text{Hom}(X^n, X^m)$ is the matrices of order $n \times m$ with binary entries (only 0 and 1) so that the amount of 1 in each row and column is at most 1.

Remark 1.2. The field with one element is a sub prop of all props generated by rings. This is good because we expect the field with one element to be the subring of all rings. Thus, the field with one element is the initial element of the category of props.

Definition 1.2 (PROP addition). Addition of PROPs is the direct block sum of the matrices. Let $A \in P_{n,m}$ and $B \in P_{k,d}$. Then

$$A \oplus B = \begin{pmatrix} A & 0_{n,d} \\ 0_{k,m} & B \end{pmatrix}.$$

Definition 1.3 (PROP multiplication). Multiplication of PROPs is given by composition. Therefore we can only multiply props of the form $A \in P_{m,n}$ and $B \in P_{n,k}$. The product of A and B is

$$A \otimes B = AB.$$

Definition 1.4 (Bioperads). Bioperads are the props $P_{1,n}$ and $P_{n,1}$ respectively. We denote $B^-(n) = P_{n,1}$ and $B^+(n) = P_{1,n}$.

Remark 1.3. We have multiple notions of commutativity of props. We also of the following inclusions.

$$\mathbf{CRing} \subseteq \mathbf{C}_T \text{Prop} \subseteq \mathbf{CProp} \subseteq \text{Prop}$$

2 Riemann Surfaces

2.1 Introduction

Definition 2.1 (Riemann surface). A Riemann surface is a 1-dimensional complex manifold.

Definition 2.2 (Riemann surface). A Riemann surface is a topological space X together with open subsets $\{U_k\}_{k \in I}$ of X with $\cup_{k \in I} U_k = X$ together with maps $f_i: U_i \rightarrow \mathbb{C}$ such that

- (1) Each f_i is a homeomorphism onto its image.
- (2) If $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1}: f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ are *biholomorphic*.

Remark 2.1. A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic at p if $f'(p) = \lim_{z \rightarrow p} \frac{f(z) - f(p)}{z - p}$ exists.

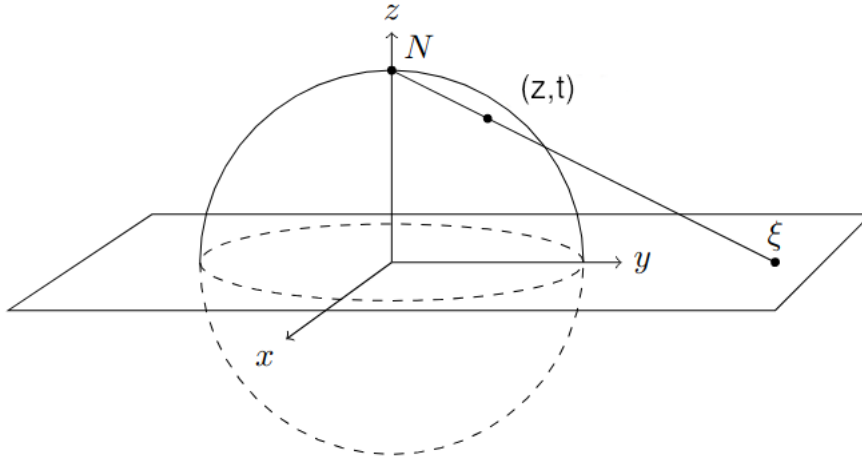
Definition 2.3 (Biholomorphism). A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is called biholomorphic if it has an inverse and both f and f' are holomorphic.

Definition 2.4 (Atlas). The $\{(U_i, f_i)\}_{i \in I}$ are called an atlas of the Riemann surface.

Definition 2.5 (Chart). Each individual (U_i, f_i) is called a chart of the Riemann surface.

Example 2.1. Let $U \subset \mathbb{C}$. Then U can take an atlas with one chart which is the identity map.

Example 2.2 (Riemann sphere). Let $X = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = \mathbb{R}\}$. We identify $\mathbb{C}$ with the xy plane. Denote N and S the north and south poles of the sphere accordingly. We define $\pi_N: \mathbb{C} \rightarrow S$ such that π_N sends each point (z, t) on the sphere to its stereographic projection from N onto the plane (point ξ) as can be seen in the figure below:



We can similarly define π_S and verify that the images of the projections are $X \setminus \{N\}$ and $X \setminus \{S\}$ accordingly.

Now X is a Riemann surface with an atlas consisting of $\pi_S: X \setminus \{S\} \rightarrow \mathbb{C}$ and $\pi_N: X \setminus \{N\} \rightarrow \mathbb{C}$. We denote the Riemann sphere as $\hat{\mathbb{C}}$.

Definition 2.6 (Biholomorphism of Riemann surfaces). Let $(X, (U_i, f_i))$, $(Y, (W_i, g_i))$ be two Riemann surfaces. A biholomorphism between them is a homeomorphism $X \xrightarrow{\phi} Y$ such that $g_i \circ \phi \circ f_i^{-1}$ are biholomorphisms on their domains of definition.

A main problem in Riemann surfaces was classifying certain types of Riemann surfaces up to biholomorphisms.

Theorem 2.1. (Riemann mapping theorem). *Any two proper open simply connected subsets of $\mathbb{C}$ are biholomorphic.*

A generalization of the Riemann mapping theorem is the uniformization theorem proved by Koebe in 1907.

Theorem 2.2. (Uniformization theorem). *Any simply connected Riemann surface is biholomorphic to one of the following:*

(1) $\mathbb{C}$

(2) $\hat{\mathbb{C}}$

(3) $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$

We will give a proof for this theorem in the end of the class.

A natural question that arises is what about non simply connected Riemann surfaces?

Theorem 2.3. (Uniformization theorem, part II). *Any connected Riemann surface is biholomorphic either to $\hat{\mathbb{C}}$ or to a quotient of $\mathbb{C}$ or $\mathbb{H}$ by a properly discontinuous torsion-free subgroup of biholomorphisms.*

Remark 2.2. Biholomorphisms of $U = \mathbb{C}$ or $\mathbb{H}$ (or any subset of $\mathbb{C}$) forms a group under composition. We denote that group by $\text{Bih}(U)$.

Definition 2.7 (Properly discontinuous group). A countable subgroup of $\text{Bih}(U)$ is said to be properly discontinuous if for all compact $K \subseteq U$, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

Definition 2.8 (torsion-free action). $G \subseteq \text{Bih}(U)$ is torsion-free if $gp = p$ for some $p \in U$ implies g is the identity.

Remark 2.3. Notice that multiplication in gp is the group action of g on the set U . That is $gp = g(p)$.

We can now define the quotient space U/G where $p \sim q$ if there exists $g \in G$ such that $gp = q$.

Introduce a topology on U/G which is the coarsest topology such that the canonical projections $U \rightarrow U/G$ are continuous.

Under the assumptions that G is properly discontinuous and torsion-free, U/G is a Riemann surface with the following charts. By assumptions on G , we can find for any $p \in U$ a neighbourhood W of $p \in U$ such that $\pi : U \rightarrow U/G$ is a homeomorphism onto its image when restricted to W .

So, restrictions of π to these neighbourhoods W provide an atlas.

Definition 2.9 (Free action). Let G

2.2 Introduction to Teichmüller spaces

Let S be a topological space. Then

$$\text{ModuliSpace}(S) = \{\text{Riemann surfaces homeomorphic to } S \text{ up to biholomorphisms}\}$$

Definition 2.10 (Teichmüller space). We consider pairs (X, f) where X is a Riemann space and $f : S \rightarrow X$ is a homeomorphism. Then

$$\text{Teich}(S) = \{(X, f) : f : S \rightarrow X\} / \sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if there exists a biholomorphism $X_1 \xrightarrow{\phi} X_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to ϕ . In other words commutes up to homotopy.

Remark 2.4. The pair (X, f) is called a *marking* for S .

Definition 2.11 (Homotopy). We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H: X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 2.12 (Mapping class group). First we define:

$$\begin{aligned} \text{Homeo}^+(S) &= \{\text{Orientation preserving homeomorphisms } S \rightarrow S\} \\ \text{Homeo}^0 \triangleleft \text{Homeo}^+(S) &= \{\text{the homeo. homotopic to the identity.}\} \end{aligned}$$

And now we define

$$\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}^0(S) = \frac{\text{Diff}^+(S)}{\text{Diff}^0(S)}.$$

We have that MCG acts on $\text{Teich}(S)$ by $\varphi \in \text{Homeo}^+(S), [\varphi] \in \text{MCG}(S)$ as such

$$[\varphi][(X, f)] = [(X, f \circ e^{-1})].$$

Now $\mathcal{M}(S) = \text{Teich}(S) / \text{MCG}(S)$. Our goal is to determine $\mathcal{M}(S)$ and $\text{MCG}(S)$ when S is a torus.

Theorem 2.4. Assume S is a torus $T \cong \mathbb{R}^2 / \mathbb{Z}^2$. Then $\text{MCG}(S) = \text{SL}_2(\mathbb{Z})$ acting linearly on the torus. $\mathcal{M}(S)$ can be identified with $\mathbb{H} / \text{SL}_2(\mathbb{Z})$ where the action is by Mobius transformations.

Proof. Any Riemann surfaces homeomorphic to S has to form $\mathbb{C} / \Lambda$ where $\Lambda \subseteq \text{Bih}(\mathbb{C})$ is properly disc, free, and $\Lambda \langle z \rightarrow z + \tau_1, z \rightarrow z + \tau_2 \rangle$ where $\tau_1, \tau_2 \notin \mathbb{R}$. We have to determine when different $\mathbb{C} / \Lambda$ are biholomorphic. We can also write $\mathbb{C} / \Lambda$ where $\Lambda = \langle \tau_1, \tau_2 \rangle \subseteq \mathbb{C}$. If $\Lambda_1 = \langle \tau_1, \tau_2 \rangle, \Lambda_2 = \langle c\tau_1, c\tau_2 \rangle$ for $c \in \mathbb{C}^*$. Then $\mathbb{C} / \Lambda_1 \rightarrow \mathbb{C} / \Lambda_2$ is given by $[z] \rightarrow [cz]$.

For any $\tau_1, \tau_2 \in \mathbb{C}$ there exists $c \in \mathbb{C}^\times$ such that (up to change in order)

This tells us that any Riemann torus is biholomorphic to $\mathbb{C} / \langle \tau X, X \rangle$ where $\tau \in \mathbb{H}$.

So we have a surjection $\mathbb{H} \rightarrow \mathcal{M}(S)$. □

Theorem 2.5. $\mathbb{C} / \langle 1, \tau_1 \rangle$ is biholomorphic to $\mathbb{C} / \langle 1, \tau_2 \rangle$ iff exists $A \in \text{SL}_2(\mathbb{Z})$ such that writing τ_1, τ_2 as elements of $\mathbb{R}^2$, we have $A\tau_1 = \tau_2$.

Proof. Suppose there exists a biholomorphism $f: \mathbb{C} / \langle 1, \tau_1 \rangle \rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle$. Let $\bar{f}: \mathbb{C} \rightarrow \mathbb{C}$ be a lift of f . This means $f(g + x) - f(x) \in \langle 1, \tau_1 \rangle$ whenever $g \in \langle 1, \tau_1 \rangle$.

$$\begin{aligned} f: \mathbb{C} / \langle 1, \tau_1 \rangle &\rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle \text{ bih.} \\ \bar{f}: \mathbb{C} &\rightarrow \mathbb{C} \text{ lift.} \end{aligned}$$

By post composing with a biholomorphism of $\mathbb{C}$ we can assume $\bar{f}(0) = 0$. We know that $\bar{f}(\tau_2)$ and $\bar{f}(1)$ are equivalent mod $\langle 1, \tau_1 \rangle$. □

Remark 2.5. Let S be a Riemann surface. Recall that $\mathcal{M}(S)$ is the moduli space of S , which is the space of Riemann surfaces homeo to S up to biholomorphism.

Recall that we define

$$\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}^0(S)$$

And the Teichmuller space of S

$$\text{Teich}(S) = \{(X, f) : f : S \rightarrow X\} / \sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if there exists a biholomorphism $X_1 \xrightarrow{\phi} X_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to ϕ .

Our goal today is to show that for $T = \text{torus} = \mathbb{R}^2 / \mathbb{Z}^2$.

- $\text{MCG}(S) \cong \text{SL}_2(\mathbb{Z})$.
- $\mathcal{M}(S) \cong \mathbb{H} / \text{SL}_2(\mathbb{Z})$.
- $\text{Teich}(S) \cong \mathbb{H}$.

Notice that because of the relation

$$\text{Teich}(S) / \text{MCG}(S) = \mathcal{M}(S)$$

we only need to prove two of these propositions.

First let's prove that $\mathcal{M}(S) \cong \mathbb{H} / \text{SL}_2(\mathbb{Z})$. Any Riemann surface is homeomorphic to T (a complex torus) is biholomorphic to $\mathbb{C} / \langle \tau_1, \tau_2 \rangle$.

Recall that if $c \in \mathbb{C}^\times$ then $\mathbb{C} / \langle \tau_1, \tau_2 \rangle$ is biholomorphic to $\mathbb{C} / \langle c\tau_1, c\tau_2 \rangle$ via the map

$$f(z + \langle \tau_1, \tau_2 \rangle) = cz + \langle c\tau_1, c\tau_2 \rangle.$$

Up to changing the order of τ_1, τ_2 we can find $c \in \mathbb{C}^\times$ such that $c\tau_1 = 1$ and $c_2 \in \mathbb{H}$.

Our goal now is to determine when there exists a biholomorphism

$$\mathbb{C} / \langle 1, \tau_1 \rangle \mapsto \mathbb{C} / \langle 1, \tau_2 \rangle, \quad \tau_1, \tau_2 \in \mathbb{H}$$

Denote $\Lambda_\tau = \langle 1, \tau \rangle$, when $\tau \in \mathbb{H}$.

Theorem 2.6. *There exists a biholomorphism*

$$\mathbb{C} / \langle 1, \tau_1 \rangle \mapsto \mathbb{C} / \langle 1, \tau_2 \rangle, \quad \tau_1, \tau_2 \in \mathbb{H}$$

iff there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

with $\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$. This will imply that

$$\mathcal{M}(S) \cong \mathbb{H} / \text{SL}_2(\mathbb{Z}).$$

Proof. Suppose there exists a biholomorphism

$$f : \mathbb{C} / \langle 1, \tau_1 \rangle \rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle.$$

Lift it up to $\bar{f} : \mathbb{C} \rightarrow \mathbb{C}$. Then $\bar{f}(g + x) - \bar{f}(x) \in \Lambda_{\tau_1}$ whenever $x \in \mathbb{C}$, $g \in \Lambda_{\tau_2}$. We can replace $\bar{f}$ with $\bar{f} - \bar{f}(0)$. We can assume that $\bar{f}(0) = 0$ (and is still a lift off).

Remark 2.6. Since $\bar{f}$ is a lift off of f we have $\bar{f}(\Lambda_{\tau_2}) = \Lambda_{\tau_1}$.

We know that $\bar{f}$ has form $\bar{f}(z) = az + b$ such that $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$. As $\bar{f}(0) = 0$ we have $\bar{f}(z) = az$. Also, $\{0, 1, \tau_2\} \subseteq \Lambda_{\tau_2}$ so $0 = \bar{f}(0)$, $\bar{f}(1) = a$, $\bar{f}(\tau_2) = a\tau_2$ are in Λ_{τ_1} so we can write $a = \bar{f}(1)$. We now have

$$\tau_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we still need to show it is an element of $\text{SL}_2(\mathbb{Z})$. □

2.3 Isotopy and Homotopy

Let X be a metric space and $F_0, F_1: X \rightarrow X$ homeomorphisms. Then we say F_0, F_1 are homotopic if there exists a family F_t for $t \in [0, 1]$ of continuous maps such that $t \mapsto F_t(x)$ is continuous for all $x \in X$ (we don't require F_0, F_1 to be homeomorphisms).

The maps F_0, F_1 are said to be isotopic if F_t are required to be homeomorphisms.

For a surface S , we defined $\text{MCG}(S)$

Theorem 2.7 (Baire, Epstein). *If S is a finite type surface (e.g. closed surface of XXX) then two homeomorphisms $F_0, F_1: X \rightarrow X$ are homotopic iff they are isotopic.*

Last time we have shown that if $T = \mathbb{R}^2/\mathbb{Z}^2$ is a torus, then $\text{MCG}(S) = \text{SL}_2(\mathbb{Z})$. For any $A \in \text{SL}_2(\mathbb{Z})$ we obtained a homeomorphism

$$\begin{aligned} \psi_A &: T \longrightarrow T \\ [x] &\longmapsto [Ax] \end{aligned}$$

We showed that any orientation preserving homeomorphism $\phi: T \rightarrow T$ is homotopic to ψ_A for some $A \in \text{SL}_2(\mathbb{Z})$.

This gives a map

$$\begin{aligned} \Phi &: \text{SL}_2(\mathbb{Z}) \longrightarrow \text{MCG}(T) \\ [A] &\longmapsto [\psi_A] \end{aligned}$$

which we know is surjective. Why is Φ injective?

We want to show that for $A \in \text{SL}_2(\mathbb{Z})$ and $A \neq I$ that ψ_A is not homotopic to the identity map.

We will see this by showing that ψ_A acts nontrivially on the fundamental group $\Pi_1(T)$.

Definition 2.13 (Loop). A loop based at p is a continuous function $\gamma: [0, 1] \rightarrow X$ with $f(0) = f(1) = 0$.

Definition 2.14 (Homotopy). We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H: X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 2.15 (Fundamental group). The fundamental group of a topological space X is the space of all loops in X based at a point $p \in X$ up to homotopy.

A homeomorphism $\psi: X \rightarrow X$ acts on $\Pi_1(X)$ by

$$\psi[\gamma] = [\psi \circ \gamma].$$

The group operation is concatenation of loops.

Inversion is changing the direction of the loop.

Exercise 2.1 (Fundamental group of the torus). The fundamental group of T is $\Pi_1(T) \cong \mathbb{Z}^2$ is generated by

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \quad \text{and} \quad \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

This means any loop is homotopic to $\left[\begin{pmatrix} a \\ b \end{pmatrix} \right]$ by doing $\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$ a times and $\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ b times.

For $A \in \mathrm{SL}_2(\mathbb{Z})$ and $\psi_A: T \rightarrow T$ action on $\Pi_1(T, (0,0))$ (fundamental group of the torus based in $(0,0)$) defined by

$$(x, y) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

If ψ_A was homotopic to the identity it would act trivially on $\Pi_1(T, (0,0))$ which means that

$$[0, 1] = [A(1, 0)] = [(b, d)] \text{ and } [(1, 0)] = [A(1, 0)] = [(a, c)]$$

so $(a, c) = (1, 0)$ and $(b, d) = (0, 1)$ so A is the identity matrix.

2.4 Hyperbolic geometry

Recall that $\mathrm{Bih}(\mathbb{H})$ is the set of Mobius transformations with a representing matrix in $\mathrm{SL}_2(\mathbb{Z})$.

But $z \mapsto \frac{1}{z}$ does not present Euclidean metric on $\mathbb{H} = \{z: \mathrm{Im}(z) > 0\}$.

We will introduce a metric $\rho = \rho_{hyp}$ on $\mathbb{H}$ s.t.

$$\mathrm{Bih}(\mathbb{H}) = \text{Orientation preserving isometrics of } (\mathbb{H}, \rho).$$

Definition 2.16 (Length). Let $\gamma: [0, 1] \rightarrow \mathbb{H}$ be a pointwise continuous smooth path between a and b .

The hyperbolic length of γ is

$$L_\rho(\gamma) := \int_0^1$$

Recall that every element in every $\mathrm{Mob}(\hat{\mathbb{C}})$ is conjugate to either $z \mapsto z + 1$ or $z \mapsto \Lambda z$ for $\Lambda \in \mathbb{C} \setminus \{0, 1\}$.

Recall that $T_\lambda(z) = \Lambda z$ and $T_{\Lambda^{-1}}(z) = \Lambda^{-1}z$ are conjugate via the transform

$$S(z) = \frac{1}{z}.$$

Indeed this can be verified.

So in the above classification we can assume $|\Lambda| \geq 1$.

Corollary 2.8. A complete set of nonidentity conjugacy class representations in $\mathrm{Mob}(\mathbb{C})$ is $z \mapsto z + 1$ or $z \mapsto \Lambda z$ for $\{\Lambda, 1/\Lambda\}$ distinct unordered pairs.

Proof. We know that $z \mapsto z + 1$ cannot be conjugate to $z \mapsto \Lambda z$ which has two fixed points in $\hat{\mathbb{C}}$.

We need to show that for different $\Lambda \neq 1$ and $|\Lambda| \geq 1$, $z \mapsto \Lambda z$ are not conjugate.

Recall that in general if $\psi \in \mathrm{Mob}(\hat{\mathbb{C}})$ then

$$\psi = \psi_A(z) = \frac{az + b}{cz + d} \text{ s.t. } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

There are two matrices representing ψ which are A and $-A$.

Since $\mathrm{PSL}_2(\mathbb{C}) \rightarrow \mathrm{Mob}(\hat{\mathbb{C}})$ defined as $[A] \mapsto \psi_A$ is an isomorphism, we know that ψ_A and ψ_B are conjugate in $\mathrm{Mob}(\hat{\mathbb{C}})$ if and only if A is conjugate to $\pm B$ in $\mathrm{SL}_2(\mathbb{C})$.

Now, if A is conjugate to B then A, B have the same eigenvalues.

Coming back to $z \mapsto \Lambda z$.

If $\Lambda_1, \Lambda_2 \in \mathbb{C} \setminus \{0, 1\}$ such that $|\Lambda_1|, |\Lambda_2| \geq 1$. $z \mapsto \Lambda z$ is represented by

$$\begin{pmatrix} \Lambda & 0 \\ 0 & 1/\Lambda \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

which has eigenvalues $(1, 1/\Lambda)$ for distinct $|\Lambda| \geq 1$ so they can't be conjugate. $\square$

2.5 Qualitative properties of Möbius transformations

The map $z \mapsto z + 1$ preserves the unique line in $\mathbb{H}$ which converges to ∞ (which is the fixed point of $z \mapsto z + 1$) in both directions. In other words, all the horizontal lines.

The map $z \mapsto \Lambda z$ preserves the geodesic in $\mathbb{H}$ which converges to the two fixed points $z \rightarrow \Lambda z$ (namely $0, \infty$).

Proposition 2.9. *Any discrete subgroup of $\mathbb{R}$ is $c\mathbb{Z}$ for $c \in \mathbb{R}$. In this case it is infinite cyclic.*

Proposition 2.10. *Any discrete subgroup of the circle S^1 is $\left\{e^{\frac{2\pi i k}{n}} : k = 0, 1, \dots, n-1\right\}$. In this case it is finite cyclic.*

Recall that for $g \in \mathrm{PSL}_2\mathbb{R}$ the commutator of g is $C_{\mathrm{PSL}_2\mathbb{R}}(g) = \{h \in \mathrm{PSL}_2\mathbb{R} : hg = gh\}$.

Theorem 2.11. *If $g \in \mathrm{PSL}_2\mathbb{R}$ is a (hyperbolic / parabolic / elliptical) then $C_{\mathrm{PSL}_2\mathbb{R}}(g)$ consists of the identity with all the (hyperbolic / parabolic / elliptical) elements which have the same fixed point set in $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.*

Remark 2.7. The requirement (hyperbolic / parabolic / elliptical) is redundant.

Recall that if g is parabolic then $C_{\mathrm{PSL}_2\mathbb{R}}(g)$ is conjugate to

$$C_{\mathrm{PSL}_2\mathbb{R}}(z \mapsto z + 1) = \{z \mapsto z + r : r \in \mathbb{R}\}.$$

If g is elliptic then $C_{\mathrm{PSL}_2\mathbb{R}}(g)$ is conjugate to

$$C_{\mathrm{PSL}_2\mathbb{R}}\left(z \mapsto \frac{(\cos \theta)z - \sin \theta}{(\sin \theta)z + \cos \theta}\right) = \left\{z \mapsto \frac{(\cos \theta)z - \sin \theta}{(\sin \theta)z + \cos \theta} : \theta \in [0, 2\pi)\right\}.$$

If g is hyperbolic ...

Theorem 2.12. *If $\Gamma \subseteq \mathrm{PSL}_2\mathbb{R}$ is discrete, and each nonidentity element of Γ has the same fixed point set then Γ is cyclic.*

Proof. If all nonidentity elements of Γ have the same fixed point set then they are either all hyperbolic, all parabolic or all elliptic.

(1) In case they are all hyperbolic,

$$\mathrm{fix}(\Gamma) = \{p, q\} \subseteq \partial\mathbb{H}$$

Then after conjugating by $T \in \mathrm{Mob}(\mathbb{H})$ with ...

□

3 Function Theory 2

3.1 Introduction

Definition 3.1 (The $H^\infty(U)$ class). Let U be an open subset of $\mathbb{C}$. We denote the class of analytic functions on U by $H(U)$ and define the following class of functions

$$H^\infty(U) := \left\{ f \in H(U) : \|f\|_\infty = \sup_{z \in U} |f(z)| < \infty \right\}.$$

Definition 3.2 (Schur class). We define the following class of functions

$$S := \{ f \in H^\infty(\mathbb{D}) : \|f\|_\infty \leq 1 \}.$$

Proposition 3.1 (Schwartz's–Pick lemma). Let $f \in S$ and let $z_1, z_2 \in S$. Then

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|.$$

Proof. To be added. □

Remark 3.1. Theorem 3.1 is also known as the invariant form of the Schwartz lemma.

Definition 3.3 (Pseudohyperbolic metric). Let $z, w \in \mathbb{D}$. We define the pseudohyperbolic metric by

$$\rho(z, w) := \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

Remark 3.2. The map ρ is invariant under Möbius transformations. That is, if ϕ is a Möbius transformation so that $\phi(w) = 0$ then we have $\rho(z, w) = |\phi(z)|$.

Remark 3.3. Let $f: \mathbb{D} \rightarrow \mathbb{D}$. Then from the Schwartz's–Pick lemma we have

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

This means that f is a contraction. Moreover, there is an equality if and only if f is an automorphism of the disc. That is because if f is an automorphism of the disc then f and f^{-1} are both contractions which implies the equality.

Problem (Pick's interpolation problem). Let $z_1, \dots, z_n \in \mathbb{D}$ and let $w_1, \dots, w_n \in \mathbb{D}$. Does there exist a function $f \in H(\mathbb{D})$ so that $f(z_i) = w_i$ for all $1 \leq i \leq n$ and $|f(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$?

Remark 3.4. In case $n = 1$ we can solve the function immediately by using $f = \psi_{w_1} \circ \psi_{z_1}$ which first interchanges between 0 and z_1 then by 0 and w_1 so $f(z_1) = w_1$ as wanted.

In case $n = 2$ we can do this if and only if $\rho(w_1, w_2) \leq \rho(z_1, z_2)$. An equivalent statement is that

$$\begin{bmatrix} \frac{1-|z_1|^2}{1-|w_1|^2} & \frac{1-\overline{z_1}z_2}{1-\overline{w_1}w_2} \\ \frac{1-\overline{z_2}z_1}{1-\overline{w_2}w_1} & \frac{1-|z_2|^2}{1-|w_2|^2} \end{bmatrix} \geq 0.$$

Theorem 3.2 (Pick's theorem). Let $z_1, \dots, z_n \in \mathbb{D}$ and let $w_1, \dots, w_n \in \mathbb{D}$. Then there exists $f \in H(\mathbb{D})$ so that $f(z_i) = w_i$ for all $1 \leq i \leq n$ if and only if the Pick matrix:

$$\left[\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive-semidefinite.

Proof. When $n = 1$ the condition is that

$$\frac{1 - w\bar{w}}{1 - z\bar{z}} \geq 0$$

which holds for any $z, w \in D$. When $n = 2$ the condition is that

$$\begin{bmatrix} \frac{1 - |w_1|^2}{1 - |z_1|^2} & \frac{1 - w_1\bar{w}_2}{1 - z_1\bar{z}_2} \\ \frac{1 - w_2\bar{w}_1}{1 - z_2\bar{z}_1} & \frac{1 - |w_2|^2}{1 - |z_2|^2} \end{bmatrix} \geq 0$$

which reduces to $\rho(w_1, w_2) \leq \rho(z_1, z_2)$. □

Definition 3.4 (Hardy space). We define the Hardy space $H^2(\mathbb{D})$ as followsing

$$H^2(\mathbb{D}) = H^2 = \left\{ h(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \text{ s.t. } \|h\|_2^2 := \sum |a_n|^2 < \infty \right\}.$$

Remark 3.5. Given a power series $\sum a_n z^n \in H^2$ we have by the Cauchy–Schwarz’s inequality

$$\left| \sum a_n z^n \right| \leq \left(\sum |a_n|^2 \right)^{1/2} \left(\sum |z|^{2n} \right)^{1/2}$$

which converges for any $z \in \mathbb{D}$.

Definition 3.5 (Inner product). Given two formal sums we define their inner product to be

$$\langle a, b \rangle = \left\langle \sum a_n z^n, \sum b_n z^n \right\rangle_{H^2} = \left\| \sum a_n \bar{b}_n \right\| \leq \|a\|_2 \|b\|_2$$

Remark 3.6. The space H^2 is a complete metric space and an inner product space so it is a Hilbert space.

Definition 3.6 (Szego kernel). The Szego kernel is a function $K: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$K(z, w) := \frac{1}{1 - z\bar{w}}$$

and the kernel function at $w \in \mathbb{D}$ is

$$k_w(z) = k(z, w) = \frac{1}{1 - z\bar{w}}.$$

Proposition 3.3. Let $k_w \in H^2$, let $w \in \mathbb{D}$ and let $h \in H^2$. Then $h(w) = \langle h, k_w \rangle$ and consequently from the Cauchy–Schwarz inequality we obtain $|h(w)| \leq \frac{1}{\sqrt{1 - |w|^2}} \|h\|_2$.

Proof. We have that

$$k_w(z) = k(z, w) = \frac{1}{1 - z\bar{w}}.$$

to be added □

Definition 3.7 (Bounded operator). Let X, Y be normed spaces and let $T: X \rightarrow Y$ be a linear function. Then T is said to be bounded if and only if $\|T\| < \infty$. We denote $B(X, Y)$ the space of bounded operators from X to Y .

Remark 3.7. The dual space of X^* is $X^* = B(X, \mathbb{C})$.

Remark 3.8. Let $w \in \mathbb{D}$ and define $ev_w: H^2 \rightarrow \mathbb{C}$ by $ev_w(h) = h(w) = \langle h, k_w \rangle$ or $h \mapsto h(w)$. The evaluation map of the Hardy space $ev_w: H^2 \rightarrow \mathbb{C}$ is bounded.

Definition 3.8 (Reproducing kernel Hilbert space). Let H be a linear subspace of the space of functions from X to $\mathbb{C}$. Then H is said to be a RKHS if H is a Hilbert space and point evaluation ev_x is bounded for all $x \in X$.

By Riesz theorem for all $\phi \in H^*$ there exists $g \in H$ so that for all $h \in H$ we have $\phi(h) = \langle h, g \rangle$. Then

Definition 3.9 (Kernel function).

Definition 3.10 (Multiplier algebra). Let H be an RKHS on X . Then the multiplier algebra on H is the space

$$\text{Mult}(H) := \{f: X \rightarrow \mathbb{C} \mid h \in H \implies fh \in H\}.$$

Question. Can we compute $\text{Mult}(H^2)$?

Answer. We know that all polynomials are in $\text{Mult}(H^2)$.

Proposition 3.4. Let H be a RKHS on X and let $f \in \text{Mult}(H)$. Then for all $x \in X$ we have $M_f^* k_x = \overline{f(x)} k_x$. Conversely, if $T \in B(H)$ then $Tk_x = \lambda_x k_x$ which implies that $T = M_f^*$ and $f \in \text{Mult}(H)$.

Corollary 3.5. Every multiplier is a bounded function

$$|f(x)| \leq \|M_f^*\| = \|M_f\|.$$

Corollary 3.6. We have that $\text{Mult}(H^2) \subseteq H^\infty(\mathbb{D})$ since $\|f\|_\infty \leq \|f\|_{\text{Mult}}$.

Proof of Theorem 3.4. Let $h \in H$ and let $f \in \text{Mult}(H)$. Then

$$\begin{aligned} f(x)h(x) &= \langle fh, k_x \rangle \\ &= \langle M_f h, k_x \rangle \\ &= \langle h, M_f^* k_x \rangle. \end{aligned}$$

□

Definition 3.11 (Multiplication algebra norm). We define for $f \in \text{Mult}(H)$

$$\|f\|_{\text{Mult}} := \|M_f\| = \sup_{h \neq 0} \frac{\|M_f h\|}{\|h\|} < \infty$$

Proposition 3.7. Let H be a RKHS on X and $t \in B(H)$. Then we have

$$\forall x \in X \quad \exists \lambda_x \in \mathbb{C} \text{ s.t. } TK_x = \lambda_x K_x \iff \exists f \in M(H) \text{ s.t. } T = M_f^*$$

and in this case $\lambda_x = \overline{f(x)}$.

Definition 3.12. Let $f: \mathbb{D} \rightarrow \mathbb{C}$ and let $r \in (0, 1)$. Then we define $f_r(z) = f(rz)$.

Remark 3.9. If $f \in H^2$ then $f_r \in H^2$. This is because if $f(z) = \sum_{n=0}^{\infty} a_n z_n$ then we have $f_r(z) = \sum_{n=0}^{\infty} r^n a_n z_n$.

Proposition 3.8. Let $f(z) = \sum_{n=0}^{\infty} a_n z_n$. Then

$$\|f\|_2^2 = \sum |a_n|^2 = \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f_r(e^{it}) \right|^2 dt.$$

Remark 3.10. We also have that

$$\|f\|_2^2 = \int_T |f|^2 dm$$

where m is the Lebesgue measure on the circle.

Proof. to be added. □

Proposition 3.9. Let $f \in H^\infty$ and $h \in H^2$. We have

$$\|fh\|_2^2 = \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f_r(e^{it}) h_r(e^{it}) \right|^2 dt \leq \|f\|_\infty^2 \|h\|_2^2$$

which implies that $f \in \text{Mult}(H)$. We also have $\|f\|_{\text{Mult}} = \|M_f\| \leq \|f\|_\infty$.

Corollary 3.10. We get that $\text{Mult}(H^2) = H^\infty$.

Theorem 3.11 (Pick's theorem). Let $z_1, \dots, z_n \in \mathbb{D}$ and let $w_1, \dots, w_n \in \mathbb{D}$. Then there exists $f \in H(\mathbb{D})$ so that $f(z_i) = w_i$ for all $1 \leq i \leq n$ if and only if the Pick matrix:

$$\left[\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive-semidefinite.

Proof. Let $z_1, \dots, z_n \in X$ and $w_1, \dots, w_n \in \mathbb{C}$. Let H be an RKHS on X so that $H = H^2$. If there exists $f \in \text{Mult}(H)$ so that

$$(1) \|f\|_{\text{Mult}} \leq 1;$$

$$(2) f(z_i) = w_i$$

then

$$\left[K(z_i, z_j)(1 - w_i \overline{w_j}) \right] \geq 0.$$

That is because

$$\|M_f^*\| = \|M_f\| = \|f\|_{\text{Mult}} \leq 1$$

...to be added The opposite direction holds if we assume that the kernel is of the form $K(z, w) = \frac{1}{1 - z\overline{w}}$. □

Theorem 3.12. We have that $f \in \text{Mult}(H)$ so that $\|f\|_{\text{Mult}} \leq 1$

Definition 3.13 (Harmonic function). Let $u \in C^2(\Omega)$ where Ω is a region in $\mathbb{C} = \mathbb{R}^2$. Then u is said to be harmonic if $\Delta u = u_{xx} + u_{yy} = 0$.

Proposition 3.13. A real u on a simply connected region Ω is harmonic if and only if there exists $f \in H(\Omega)$ so that $u = \text{Re}(f) = \text{Im}(if)$

Proof. Let $f = u + iv \in H(\Omega)$. From the Cauchy–Riemann equations we get $u_x = v_y$ and $v_x = -u_y$ which means that $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$.

Next suppose that u is harmonic. Define

$$V(x, y) := \int v_x dx + v_y dy = \int_\Gamma -u_y dx + u_x dy.$$

Using Green's theorem we get that

$$\int_{\partial\Omega} -u_y dx + u_x dy = \int_{\Omega} \underbrace{(u_{xx} + u_{yy})}_{=0} dx dy = 0$$

After integrating V we get that

$$\frac{\partial V}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} -u_y(t, y) dt = -u_y|_{(x, y)}$$

which completes the proof. □

Corollary 3.14. *Harmonic functions are C^∞ .*

Proof. □

Theorem 3.15. *Let $f \in \mathcal{L}^1(T)$ and let Hf be defined on the closed unit disc $\overline{\mathbb{D}}$ by*

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}), & r = 1 \\ P[f](re^{i\theta}), & 0 \leq r < 1 \end{cases}.$$

Then Hf is harmonic in $\mathbb{D}$ and $Hf \in C(\overline{\mathbb{D}})$

Remark 3.11. This solves the Dirichlet's problem.

Proof. □