

Functional Analysis

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Winter 2026

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1	Introduction	3
1.1	Linear operators	5
2	The Hahn–Banach theorem	8

1 Introduction

Definition 1.1 (Normed space). Let V be a vector space over a field $\mathbb{F}$ (usually $\mathbb{R}$ or $\mathbb{C}$). A norm $\|\cdot\| : V \rightarrow \mathbb{R}$ is a function that satisfies the following properties:

- (1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every $\lambda \in \mathbb{F}$ and $x \in V$.
- (3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

The norm $\|\cdot\|$ induces a metric $d(x, y) := \|x - y\|$ on V .

Definition 1.2 (Banach space). If (V, d) is a complete normed space, we say that $(V, \|\cdot\|)$ is a Banach space.

Example 1.1 (Hilbert space). Every Hilbert space is a Banach space where the norm is induced by an inner product.

Example 1.2 (ℓ_p spaces). Fix $1 \leq p < \infty$ and a natural number $n \geq 1$. The space ℓ_p^n is the space $\mathbb{F}^n$ with the p -norm:

$$\|x\|_p := \|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Similarly, the space ℓ^p is defined by:

$$\ell_p := \left\{ (x_n)_{n=1}^\infty \left| \sum_{i=1}^\infty |x_i|^p < \infty \right. \right\},$$

with the norm:

$$\|x\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{1/p}.$$

Finally, we define the space:

$$\ell_\infty := \left\{ (x_n)_{n=1}^\infty \left| \sup_n |x_n| < \infty \right. \right\},$$

with the norm $\|x\|_\infty := \sup_n |x_n|$.

Remark 1.1. The space ℓ^2 is a Hilbert space, where the norm is induced by the inner product $\langle x, y \rangle := \sum_{n=1}^\infty x_n \overline{y_n}$. This is not true in general for ℓ_p spaces.

Remark 1.2. For all $1 \leq p < \infty$ the space ℓ_p is separable, however ℓ_∞ is a “very big” space, and it is not separable.

Example 1.3 (C and C_0). Define:

$$C := \left\{ (x_n)_{n=1}^\infty \left| \lim_{n \rightarrow \infty} x_n \text{ exists} \right. \right\},$$

with the norm $\|x\| = \sup_n |x_n|$. The space C is a subspace of ℓ_∞ , and in particular, it is separable.

Next define:

$$C_0 := \left\{ (x_n)_{n=1}^\infty \left| \lim_{n \rightarrow \infty} x_n = 0 \right. \right\} \subseteq C \subseteq \ell_\infty.$$

Remark 1.3. A subset of a Banach space is a Banach subspace if and only if it is closed.

Example 1.4 (The space $C(X)$). Let X be a compact topological space. We define the space:

$$C(X) := \{f: X \rightarrow \mathbb{F} \mid f \text{ is continuous}\},$$

and the norm is $\|f\| := \|f\|_\infty = \max_{x \in X} |f(x)|$.

Question. What does it mean that $f_n \xrightarrow{n \rightarrow \infty} f$ in the Banach space $C(X)$?

Answer. It means that $\|f_n - f\| \xrightarrow{n \rightarrow \infty} 0$, or similarly $\max |f_n - f| \xrightarrow{n \rightarrow \infty} 0$, which is equivalent to the condition $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly.

Example 1.5 (Nonexample of a Banach space). Define the vector space:

$$C_C(\mathbb{R}^d) := \left\{f: \mathbb{R}^d \rightarrow \mathbb{R} \mid f \text{ is cont. and compactly supported}\right\},$$

and consider the norm $\|f\| := \max |f|$. This is a normed space that is *not* complete, and therefore this is not a Banach space.

For example, suppose that $d = 1$ and take $g(x) = e^{-x^2}$. Define the function sequence $\{f_n\}_{n=1}^\infty$ by:

$$f_n(x) := \begin{cases} e^{-x^2} & x \in [-n, n] \\ 0 & x \notin (-n-1, n+1) \\ \text{linear} & \text{otherwise} \end{cases}.$$

Clearly $\{f_n\}_{n=1}^\infty \subseteq C_C(\mathbb{R})$, and $\|f_n - g\|_\infty \xrightarrow{n \rightarrow \infty} 0$, so since it converges, it is a Cauchy sequence, however, the limit g is not in $C_C(\mathbb{R})$, which implies that $C_C(\mathbb{R})$ is not a Banach space.

Remark 1.4. Remember that every noncomplete metric space is dense in some other space and can be completed. It turns out that the completion of $C_C(\mathbb{R}^d)$ is the space

$$C_0(\mathbb{R}^d) := \left\{f: \mathbb{R}^d \rightarrow \mathbb{R} \mid f \text{ is cont. and } \lim_{\|x\| \rightarrow \infty} f(x) = 0\right\},$$

with the norm $\|f\| := \max |f|$.

Definition 1.3 ($\mathcal{L}^p$ spaces). Fix a measure space $(X, \mathcal{F}, \mu)$ and a constant $1 \leq p < \infty$. Define:

$$\mathcal{L}^p(X, \mathcal{F}, \mu) := \left\{f: X \rightarrow \mathbb{R} \mid f \text{ measurable and } \left(\int_X |f|^p\right)^{1/p} < \infty\right\} / \sim$$

with the norm $\|f\|_p := \left(\int_X |f|^p\right)^{1/p}$, where functions are identified by $f \sim g$ if and only if $f =_{\text{a.e.}} g$ (equal almost everywhere).

Remark 1.5. Without the equivalence relation in Definition 1.3 we could have functions which are not the zero function with norm 0.

Remark 1.6. If $X = \mathbb{N}$ and the measure is the counting measure $\mu(A) = |A|$, then $\mathcal{L}^p(X, \mu)$ is the previously defined ℓ_p space.

Question. We can convince ourself that $\mathcal{L}^p$ spaces are normed spaces, but why are they complete?

Exercise 1.1. A normed space is complete if and only if every absolutely converging series converges conditionally.

Theorem 1.1. *The space $\mathcal{L}^p(X, \mu)$ is complete.*

Proof. Fix $\{f_n\}_{n=1}^\infty \subseteq \mathcal{L}^p(\mu)$ such that $\sum_{n=1}^\infty \|f_n\| < \infty$.

First assume that $\{f_n\}_{n=1}^\infty$ is a sequence of nonnegative functions. Define the function $g(x) := \sum_{n=1}^\infty f_n(x)$, and the sequence $g_N(x) := \sum_{n=1}^N f_n(x)$. Note that

$$\|g_N\|_{\mathcal{L}^p} \leq \sum_{n=1}^N \|f_n\| \leq \underbrace{\sum_{n=1}^\infty \|f_n\|}_{:= S} < \infty.$$

Note that $g_N \nearrow g$ pointwise almost everywhere. Hence, by MCT we get that

$$\int g^p d\mu = \lim_{n \rightarrow \infty} \int g_n^p d\mu = \lim_{n \rightarrow \infty} \|g_n\|_p^p \leq S^p < \infty.$$

Therefore $g \in \mathcal{L}^p(\mu)$ and in particular $g < \infty$ almost everywhere. Now check using MCT/DCT that $\|g_N - g\|_p \xrightarrow{N \rightarrow \infty} 0$.

For a general f_n we can decompose it to its positive and negative parts:

$$f_n = \alpha_n + i\beta_n = (\alpha_n^+ - \alpha_n^-) + i(\beta_n^+ + \beta_n^-),$$

we take $|\alpha_n^\pm|, |\beta_n^\pm| \leq |f_n|$, so they also satisfy that they are nonpositive/nonnegative and satisfy

$$\sum_{n=1}^\infty |\alpha_n^\pm| < \infty \quad \text{and} \quad \sum_{n=1}^\infty |\beta_n^\pm| < \infty,$$

so $\sum_{n=1}^\infty f_n$ is a sum of four convergent sequences in and thus converges (in $\mathcal{L}^p$). $\square$

Definition 1.4 (The essential supremum). Let $(X, \mathcal{F}, \mu)$ be a measure space, and let g be a measurable function. We define:

$$\text{esssup } g := \inf \left\{ m \mid \mu \left(\{x \mid g(x) > m\} \right) = 0 \right\}.$$

Definition 1.5 ($\mathcal{L}^\infty$). We define the set $\mathcal{L}^\infty$ to be the space:

$$\mathcal{L}^\infty(X, \mathcal{F}, \mu) = \{f : X \rightarrow \mathbb{F} \mid f \text{ measurable and } \text{esssup } |f| < \infty\},$$

with the norm $\|f\|_\infty = \text{esssup } |f|$.

Exercise 1.2. Let $f \in \mathcal{L}^p \cap \mathcal{L}^\infty$ for some $1 \leq p < \infty$. Then $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$.

1.1 Linear operators

Definition 1.6. Let X, Y be normed spaces over a field $\mathbb{F}$. A linear map $T : X \rightarrow Y$ is called bounded if

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty.$$

Remark 1.7. We denote the space of bounded operators by $B(X, Y)$.

Remark 1.8. Recall that the following conditions for are equivalent for a linear operator T :

- (1) T is bounded.
- (2) T is continuous.

- (3) T is continuous at a point.
- (4) T is uniformly continuous.
- (5) T is Lipschitz continuous.

Example 1.6 (Linear noncontinuous operator). Let $\{v_i\}_{i \in I \subset \mathbb{R}}$ be a noncountable basis of ℓ_2 (in the sense of linear algebra), which exists by Zorn's lemma. We can define $T: \ell_2 \rightarrow \ell_2$ be the linear map $T(v_i) = |i|^{1000} \cdot v_i$.

Proposition 1.2. *Let Y be a Banach space. Then $B(X, Y)$ is a Banach space.*

Definition 1.7 (Dual space). Let X be a Banach space. We define the dual space of X by:

$$X^* := B(X, \mathbb{F}),$$

which is Banach from the previous theorem.

Theorem 1.3 (Riesz's representation theorem). *Let H be a Hilbert space. Then every $f \in H^*$ is of the form $f_y(x) = \langle x, y \rangle$ for some $y \in H$ and $\|y\|_H = \|f_y\|_{H^*}$.*

Remark 1.9. We have that $H^* \cong H$ by the map $y \mapsto f_y$.

Corollary 1.4 (Radon–Nikodym theorem). *Let μ, ν be (finite) measures on $(X, \mathcal{F})$. Suppose that $\mu(A) = 0$ implies $\nu(A) = 0$. Then there exists $g \in \mathcal{L}^1(\mu)$ such that $g \geq 0$ and for every $A \in \mathcal{F}$ we have $\nu(A) = \int_A g d\mu$.*

Proof. Let $H = \mathcal{L}^2(\mu + \nu)$. Define the map $l: H \rightarrow \mathbb{R}$ by $l(f) = \int_X f d\mu$.

We see that l is bounded by the Cauchy–Schwartz inequality:

$$\begin{aligned} |l(f)| &\leq \int_X 1 \cdot |f| d\mu \leq \sqrt{\int_X 1^2 d\mu} \cdot \sqrt{\int_X |f|^2 d\mu} \leq \sqrt{\mu(X)} \sqrt{\int_X |f|^2 d(\mu + \nu)} \\ &= \sqrt{\mu(X)} \|f\|_{\mathcal{L}^2(\mu + \nu)}. \end{aligned}$$

By Riesz's theorem there exists $h \in H$ such that for all $f \in H$ we have:

$$\begin{aligned} \int_X f d\mu = l(f) &= \langle f, h \rangle = \int_X f h d(\mu + \nu) = \int_X f(1 - h) d\mu \\ &= \int_X f h d\nu. \end{aligned}$$

Take $A = \{x \mid h(x) \leq 0\}$, and $f = \mathbb{1}_A \in H$. We know that

$$\mu(A) = \int_A d\mu = \int_A (1 - h) d\mu = \int_A h d\nu$$

□

Remark 1.10. Note that the inner product is simply:

$$\langle f, g \rangle = \int_X f \bar{g} d(\mu + \nu) = \int_X f \bar{g} d\mu + \int_X f \bar{g} d\nu,$$

and $\|f\|_{h^2} = \sqrt{\langle f, f \rangle}$.

Theorem 1.5. Fix $1 \leq p < \infty$. We have that $(\ell_p)^* = \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Additionally, every $f \in (\ell_p)^*$ is of the form:

$$f_c((a_n)_{n=1}^\infty) = \sum_{n=1}^\infty a_n c_n$$

for some sequence $c = (c_n)_{n=1}^\infty \in \ell_q$. Moreover, $\|f_c\|_{(\ell_p)^*} = \|c\|_{\ell_q}$.

Proof. Assume that $1 < p < \infty$. For all $c \in \ell_q$ and $a \in \ell_p$ we have

$$|f_c(a)| = \left| \sum_{n=1}^\infty a_n c_n \right| \leq \sum_{n=1}^\infty |a_n| |c_n| \leq \text{Hölder} \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p} \left(\sum_{n=1}^\infty |c_n|^q \right)^{1/q} \leq \|c\|_{\ell_q} \|a\|_{\ell_p},$$

so f_c is well-defined, bounded, and $\|f_c\|_{(\ell_p)^*} \leq \|c\|_{\ell_q}$.

In order to show equality □

Example 1.7. Let X be a compact Hausdorff topological space (for example $X = [0, 1]$). Consider the dual space $C(X)^*$.

Given a finite Borel measure μ on X , we can define the linear operator:

$$\Phi_\mu(f) = \int f d\mu \in C(X)^*.$$

Since we have

$$|\Phi_\mu(f)| = \left| \int f d\mu \right| \leq \mu(X) \cdot \|f\|_\infty,$$

we get that $\|\Phi_\mu\|_{C(X)^*} \leq \mu(X)$ so it is a bounded linear functional. Actually, there is equality for example if $f \equiv 1$.

Definition 1.8 (Complex measure). A complex measure on a measurable space $(X, \mathcal{F})$ is a function $\mu: \mathcal{F} \rightarrow \mathbb{C}$ such that for all $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ that are pairwise disjoint we have $\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.

Remark 1.11 (Hahn decomposition theorem). Every complex measure can be written as:

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4),$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are positive measures, and then we define:

$$\int f d\mu = \int f d\mu_1 - \int f d\mu_2 + \int f d\mu_3 - \int f d\mu_4.$$

Theorem 1.6 (Riesz–Markov–Kakutani representation theorem). We have that

$$C(X)^* = M(X) = \{\text{complex Borel measures on } X\}.$$

Moreover, we have that:

$$\|\Phi_\mu\|_{C(X)^*} = \|\mu\|_{M(X)} = TV(\mu) = |\mu| = \sup \left\{ \sum_{i=1}^n |\mu(A_i)| \mid X = \bigsqcup_{i=1}^n A_i \right\}.$$

Example 1.8. In a finite (or σ -finite) measure space $(X, \mathcal{F}, \mu)$, every $\Phi \in \mathcal{L}^p(X)^*$ is of the form:

$$\Phi_g(f) = \int_X f g d\mu$$

for $g \in \mathcal{L}^q(X)$ and $\|\Phi\|_{\mathcal{L}^p(X)^*} = \|g\|_{\mathcal{L}^q}$.

Proof sketch: By Hölder's inequality $\Phi_g \in (\mathcal{L}^p)^*$ and $\|\Phi_g\| \leq \|g\|_{\mathcal{L}^q}$, and like in the case of ℓ_p , this is actually an equality.

In order to show that Φ_g are all the bounded linear functionals use Radon–Nikodym (proof omitted in class).

Equivalently, there is an isometry $(\mathcal{L}^p(X))^* \cong \mathcal{L}^q(X)$ for $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.9 (Isometry). A bounded linear functional $T \in \mathcal{L}(X, Y)$ is called an isometry, if T is a bijection and

$$\|Tx\|_Y = \|x\|_X$$

and we write $X \cong Y$.

Definition 1.10 (Isomorphism). A map $T \in \mathcal{L}(X, Y)$ is an isomorphism if T is a bijection and $T^{-1} \in \mathcal{L}(Y, X)$. Equivalently, for every $T \in \mathcal{L}(X, Y)$ we have

$$c\|x\| \leq \|Tx\| \leq C\|x\|,$$

for some constants c and C .

Example 1.9. Let $X = \ell_2$ be with the usual norm, and let $Y = \ell_2$ with the norm

$$\|(a_n)\|_Y = \|(a_n)\|_{\ell_2} + |a_1|.$$

We get that X and Y are isomorphic. It follows that Y is complete (since X is complete).

2 The Hahn–Banach theorem

Remark 2.1. We claimed that $\ell_1 \subsetneq (\ell_\infty)^*$. How can we find all the elements in $(\ell_\infty)^*$ that are not in ℓ_1 ?

Definition 2.1 (Sublinear function). Let X be a vector space over $\mathbb{R}$. A function $p: X \rightarrow \mathbb{R}$ is said to be sublinear if $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for every $x, y \in X$ and $\lambda \geq 0$.

Remark 2.2. Although sublinear functions look like norms, they are certainly very different. For example, some sublinear function do not satisfy $p(x) = p(-x)$.

Theorem 2.1 (The Hahn–Banach theorem, version 1). *Let X be a vector space over $\mathbb{R}$, let $p: X \rightarrow \mathbb{R}$ be a sublinear function, let $Y \subseteq X$ be a subspace, and let $l: Y \rightarrow \mathbb{R}$ be a linear function. If $l(y) \leq p(y)$ for all $y \in Y$, then there exists a linear extension $\tilde{l}: X \rightarrow \mathbb{R}$ such that $\tilde{l}|_Y = l$ and $\tilde{l}(x) \leq p(x)$ for all $x \in X$.*

Proof. Assume that $X = \text{span}\{Y, x_0\}$ for some $x_0 \in X$, so that every $x \in X$ can be written as $x = y + \lambda x_0$ for some $y \in Y$ and $\lambda \in \mathbb{R}$.

Note that

$$\tilde{l}(x) = \tilde{l}(y + \lambda x_0) = \tilde{l}(y) + \lambda \tilde{l}(x_0) = l(y) + \lambda \tilde{l}(x_0),$$

so we only get to choose $a = \tilde{l}(x_0)$. Can we choose a such that $l(y) + \lambda a = \tilde{l}(y + \lambda x_0) \leq p(y + \lambda x_0)$.

For $\lambda = 0$ any value of a is fine, for $\lambda > 0$ we can solve for a from $l(y) + \lambda a = \tilde{l}(y + \lambda x_0) \leq p(y + \lambda x_0)$ and get that for every $y \in Y$ we need $a \leq \frac{p(y + \lambda x_0) - l(y)}{\lambda}$, and similarly we can find restriction for when $\lambda < 0$, and conclude that the restrictions are:

$$\frac{l(z) - p(z + \lambda x_0)}{\lambda} \leq a \leq \frac{p(y + \lambda x_0) - l(y)}{\lambda},$$

or equivalently, we need to show that exists a such that

$$\sup_{\mu > 0, z \in Y} \{l(z/\mu) - p(z/\mu - x_0)\} \leq a \leq \sup_{\lambda > 0, y \in Y} \{p(y/\lambda + x_0) - l(y/\lambda)\}$$

In the general case, if we need to extend Y to X by a finite amount of variables, we repeat this process. In the infinite case, we need to use Zorn's lemma. Consider:

$$\mathcal{C} = \left\{ (Z, \tilde{l}) \mid \tilde{l}|_Y = \ell \text{ and } \tilde{l} \leq p \text{ on } Z \right\},$$

for $Z \subseteq Y \subseteq X$. We can order $\mathcal{C}$ by saying that $(Z_1, \tilde{l}_1) \leq (Z_2, \tilde{l}_2)$ if $Z_1 \subseteq Z_2$ and $\tilde{l}_2|_{Z_1} = \tilde{l}_1$.

Take $\{(Z_i, \tilde{l}_i)\}_{i \in I}$ to be a chain in $\mathcal{C}$. Take $Z = \bigcup_{i \in I} Z_i$ and l be the “obvious” extension (every $z \in Z$ belongs to some Z_i , take $l(z) = \tilde{l}_i(z)$, since they agree by the chain containment). Thus, (Z, l) is an upper bound for the chain. By Zorn's lemma $\mathcal{C}$ has a maximal element, (Z, ℓ) . By step 1, $Z = X$. $\square$

Theorem 2.2 (The Hahn–Banach theorem, version 2). *Let X be a normed space over (for example a Banach space), let $Y \subseteq X$ be a subspace, and let $l \in Y^*$ be a continuous functional. Then there exists some linear functional $\tilde{l} \in X^*$ such that $\tilde{l}|_Y = l$ and $\|\tilde{l}\|_{X^*} = \|l\|_{Y^*}$.*

Example 2.1. Let ℓ_∞ and $Y = C$. Take $f \in Y^*$ to be the functional defined by

$$f((a_n)_{n=1}^\infty) = \lim_{n \rightarrow \infty} a_n.$$

Note that

$$\left| f((a_n)) \right| = \left| \lim_{n \rightarrow \infty} a_n \right| = \lim_{n \rightarrow \infty} |a_n| \leq \sup_n |a_n| = \|a\|_Y,$$

it follows that $\|f\|_{Y^*} \leq 1$. However, we notice that for $a = (1, 1, 1, \dots)$ we see $\|f\|_{Y^*} = 1$.

By the Hahn–Banach theorem, we can extend f to $\tilde{f} \in (\ell_\infty)^*$ such that $\|\tilde{f}\|_{(\ell_\infty)^*} = 1$.

Note that $\tilde{f}$ is not of the form:

$$\tilde{f}((a_n)_{n=1}^\infty) = \sum_{n=1}^\infty a_n b_n \quad (b_n) \in \ell_1,$$

since otherwise, $b_n = \tilde{f}(e_n) = f(e_n) = 0$. Thus $b_n = 0$ for all n and then $\tilde{f} \equiv 0$ in contradiction to $\|\tilde{f}\|_{Y^*} = 1$.

Proof of Theorem 2.2. Choose $p(x) = \|f\|_{Y^*} \|x\|_X$. Clearly, $p(x)$ is sublinear. On Y , we have $l(y) \leq p(y) = \|l\|_{Y^*} \|y\|$. By version 1 of the Hahn–Banach theorem, there exists an extension $\tilde{l}$ of l such that $\tilde{l}(x) \leq p(x) = \|l\|_{Y^*} \|x\|$ for all $x \in X$. But then also

$$-\tilde{l}(x) = \tilde{l}(-x) \leq p(-x) = \|l\|_{Y^*} \|x\|,$$

and thus

$$|\tilde{l}(x)| \leq \|l\|_{Y^*} \|x\| \implies \|\tilde{l}\|_{X^*} \leq \|l\|_{Y^*}.$$

The inequality $\|\tilde{l}\|_{X^*} \geq \|l\|_{Y^*}$ is true for all extensions. This completes the proof for real normed spaces.

For the case of complex normed spaces, we decompose l into its real and complex parts, and use some algebra and the proof of the real case in order to complete the proof. $\square$

Corollary 2.3. *Let X be a Banach space. Then for every $x \in X$ there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof. Let $Y = \text{span } x \subseteq X$, and define $f: Y \rightarrow \mathbb{F}$ by $f(\lambda x) = \lambda \|x\|$.

We have that f is linear and $\|f\| = 1$ and

$$|f(\lambda x)| = |\lambda| \|x\| = \|\lambda x\|.$$

Extend f by the Hahn–Banach theorem to conclude the proof. $\square$

Corollary 2.4. *For all $x \neq y \in X$ there exists $f \in X^*$ such that $f(x) \neq f(y)$.*

Proof. By Theorem 2.3 take f with $\|f\| = 1$ and then:

$$f(x) - f(y) = f(x - y) = \|x - y\| \neq 0.$$

□

Corollary 2.5. *Let X be a normed space, let $E \subseteq X$ be a subspace, and let $x_0 \in X$ be a point such that $d(x_0, E) = d > 0$. Then there exists $f \in X^*$ such that $\|f\| = 1$, and $f(E) = 0$, and $f(x_0) = d$.*

Proof. Take $Y = \text{span}\{E, x_0\}$. Define $f \in Y^*$ by $f(e + \lambda x_0) = \lambda d$.

For all $e \in E$ and $\lambda \in \mathbb{F}$

$$\|e + \lambda x_0\| = \left\| \lambda \left(x_0 - \left(-\frac{e}{\lambda} \right) \right) \right\| = |\lambda| \cdot \left\| x_0 - \underbrace{\left(-\frac{e}{\lambda} \right)}_{\in E} \right\| \geq |\lambda| d = |f(e + \lambda x_0)|,$$

it follows that $\|f\|_{Y^*} \leq 1$ (in fact, there is an equality). We now extend by Hahn–Banach to complete the proof. □