

Complex Analysis

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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1 Introduction

1.1 Complex numbers and the complex plane

1.1.1 Preliminaries

Definition 1.1 (Complex number). A complex number is an expression of the form $x + yi$ such that $x, y \in \mathbb{R}$ and i is a ‘imaginary number’ not in $\mathbb{R}$. We denote

$$\operatorname{Re}(z) := x \quad \text{and} \quad \operatorname{Im}(z) := y.$$

If $\operatorname{Im}(z) = 0$ then z is said to be a real number, and if $\operatorname{Re}(z) = 0$ then it is said to be purely imaginary.

Remark 1.1. The set of all complex numbers is denoted as $\mathbb{C}$ and it can be made into a field with the following operations.

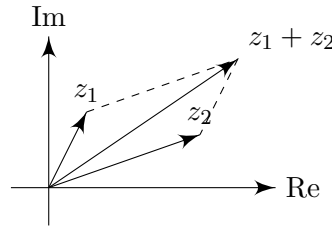
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field $\mathbb{C}$ is called the complex plane.

Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between $\mathbb{C}$ and $\mathbb{R}^2$ and moreover, we have that T is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

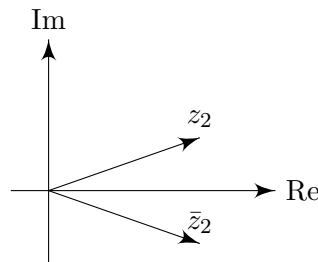
$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \|(x, y)\| = \|T(z)\|$ where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

This implies that $|z - w|$ should be considered the distance between natural numbers z, w . Because we have that $|z| = \|T(z)\|$ we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \quad \text{for all } z, w \in \mathbb{C}.$$

Definition 1.2 (Complex conjugate). The complex conjugate of $x + yi = z \in \mathbb{C}$ is the complex number $x - yi$. The complex conjugate of z is denoted $\bar{z}$.



Remark 1.2. It is easy to verify that

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given θ we can denote $e^{i\theta} = \cos \theta + i \sin \theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that θ describes the angle of z with the x -axis and it is usually denoted $\theta = \arg(z)$.

1.1.2 Convergence

Definition 1.3 (Convergence). We say that the sequence $\{z_n\}_{n \geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n \rightarrow \infty} 0$. In this case, we call z_0 the limit of the sequence of $\{z_n\}_{n \geq 1}$.

Remark 1.3. It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n \rightarrow \infty} z$ if and only if $T(z_n) \xrightarrow{n \rightarrow \infty} T(z)$ in the Euclidean metric.

Definition 1.4 (Cauchy sequence). A sequence $\{z_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

Proposition 1.1. *The complex plane $\mathbb{C}$ is complete. That is, every Cauchy sequence converges in $\mathbb{C}$.*

Proof. The proof follows immediately from the known fact that $\mathbb{R}$ is complete and the previous remark. $\square$

1.1.3 Sets in the complex plane

Definition 1.5 (Open disc). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We call $D_r(z_0)$ the open disc at center z_0 with radius r .

Definition 1.6 (Closed disc). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center z_0 with radius r .

Definition 1.7 (Circle). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}.$$

We call $C_r(z_0)$ the circle at center z_0 with radius r .

Definition 1.8 (Interior point). Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if exists $r > 0$ such that $D_r(z) \subset \Omega$.

Definition 1.9 (Interior of a set). Given $\Omega \subset \mathbb{C}$, we say that the interior of Ω is the collection of all interior points of Ω . We denote the interior of Ω by $\operatorname{Int}(\Omega)$.

Definition 1.10 (Open set). Given $\Omega \subset \mathbb{C}$, we say that Ω is an open set if $\operatorname{Int}(\Omega) = \Omega$.

Definition 1.11 (Closed set). Given $\Omega \subset \mathbb{C}$, we say that Ω a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

Definition 1.12 (Limit point). Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is a limit point of Ω if there exists a sequence $\{z_n\}_{n=1}^\infty \subset \Omega$ such that $z_n \neq z$ for all $n \geq 1$ and $z_n \xrightarrow{n \rightarrow \infty} z$.

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be given. Then Ω is closed if and only if it contains all of its limit points.*

Proof. Suppose that Ω is not closed. Then $\mathbb{C} \setminus \Omega$ is not open, which implies that there exists $z \in \mathbb{C} \setminus \Omega$ such that for all $n \geq 1$ there exists $z_n \in \Omega \cap D_{1/n}(z)$. Since it is clear that $z \neq z_n \xrightarrow{n \rightarrow \infty} z$, we get that z is a limit point of Ω which is not contained in Ω . This completes one direction of the proposition.

Next suppose that there exists $z \in \mathbb{C} \setminus \Omega$ which is a limit point of Ω . Thus it can't be an interior point of $\mathbb{C} \setminus \Omega$, which means that $\mathbb{C} \setminus \Omega$ is not open, so Ω is not closed which completes the proof. $\square$

Definition 1.13 (Closure). Let $\Omega \subset \mathbb{C}$ be given. The closure of Ω , denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \{z \in \mathbb{C} \mid z \text{ is a limit point of } \Omega\}.$$

Remark 1.4. Note that Ω is closed if and only if $\overline{\Omega} = \Omega$.

Definition 1.14 (Boundary). The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \overline{\Omega} \setminus \text{Int}(\Omega)$.

Definition 1.15 (Diameter). Given $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\text{diam}(\Omega) := \sup \{|z - w| : z, w \in \Omega\}.$$

Definition 1.16 (Bounded set). Given $\Omega \subset \mathbb{C}$, we say that Ω is bounded if $\text{diam}(\Omega) < \infty$.

Remark 1.5. It is clear that a set $\Omega \subset \mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega \subset D_r(z_0)$.

Definition 1.17 (Compact set). A subset Ω of $\mathbb{C}$ is said to be compact if it is closed and bounded.

Theorem 1.3 (Bolzano–Weierstrass theorem). *A subset Ω in $\mathbb{C}$ is compact if and only if every sequence $\{z_n\}_{n \geq 1}$ has a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \xrightarrow{k \rightarrow \infty} z$ for some $z \in \mathbb{C}$.*

Remark 1.6. Another formulation of the Bolzano–Weierstrass theorem is that a subset of $\mathbb{R}^n$ (equivalently of $\mathbb{C}$) is sequentially compact if and only if it is closed and bounded. This is why it is also occasionally called the sequential compactness theorem.

Theorem 1.4 (Cantor's intersection lemma). *Let $\Omega_1, \Omega_2, \dots$ be nonempty compact subsets of $\mathbb{C}$. Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that $\text{diam}(\Omega_n) \xrightarrow{n \rightarrow \infty} 0$. Then $\bigcap_{n \geq 1} \Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

Proof. Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\text{diam}(\Omega_n) \xrightarrow{n \rightarrow \infty} 0$ we have that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because Ω_n is compact for every $n \geq 1$ we get that $z \in \bigcap_{n \geq 1} \Omega_n$. This means that $\bigcap_{n \geq 1} \Omega_n \neq \emptyset$.

Let $z, w \in \bigcap_{n \geq 1} \Omega_n$. Because $\text{diam} \Omega_n \xrightarrow{n \rightarrow \infty} 0$ we have that $|z - w| \leq 0$ and thus $z = w$ which implies that $\bigcap_{n \geq 1} \Omega_n = \{z\}$ which completes the proof. $\square$

Definition 1.18 (Connected open set). A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Definition 1.19 (Region). A connected open set in $\mathbb{C}$ is called a region.

Definition 1.20 (Connected closed set). A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1.7. It can be shown that Ω is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma: [0, 1] \rightarrow \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$. This implies that open and closed discs, as well as circles, are connected.

1.1.4 Continuous functions

Definition 1.21 (Continuous function). Let Ω be a nonempty subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that f is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Remark 1.8. It is easy to verify that the functions Im , Re , $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

Proposition 1.5. *The addition, product, and composition of continuous functions is continuous.*

Definition 1.22 (Bounded function). Let Ω be a nonempty subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that f attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when f attains a minimum similarly.

Proposition 1.6. *Let Ω be a nonempty compact subset of $\mathbb{C}$, and let $f: \Omega \rightarrow \mathbb{C}$ be continuous. Then f is bounded, and it attains its maximum and minimum on Ω .*

1.2 Holomorphic functions

Definition 1.23 (Holomorphic function). Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of f at z . It is said that f is holomorphic if it is holomorphic at every $z \in \Omega$. Given a closed subset $C \subset \Omega$, we say that f is holomorphic on C if there exists $C \subset \Omega' \subset \Omega$ so that Ω' is open and f is holomorphic on Ω' .

Remark 1.9. Note that h is a complex number and can approach 0 from any direction.

Remark 1.10. This is not a course in computations of complex derivatives. While it is important to know Definition 1.23, there is no need to be able to visualize it like in analysis 1. Instead we use simple tricks and complex differentiation rules that can be seen in the problems section.

Remark 1.11. As we mentioned earlier, given a complex function in the form $f(x, y) = u(x) + iv(y)$ we can also represent it as a function by $f(z, \bar{z})$ and thus it makes sense to define partial derivatives with respect to z and $\bar{z}$. However, unlike partial derivatives with respect to x or y , the most natural definition (the Wirtinger derivative) may seem a little pathological.

Definition 1.24 (Wirtinger derivative). We define the Wirtinger derivative as

$$\begin{aligned} \frac{\partial f}{\partial z} &:= \frac{1}{2} \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \bar{z}} &:= \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y}. \end{aligned}$$

Definition 1.25 (Entire function). We say that $f: \mathbb{C} \rightarrow \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$.

Remark 1.12. It is also useful to notice that $f: \Omega \rightarrow \mathbb{C}$ is holomorphic at $z \in \Omega$ if and only if there exist $a \in \mathbb{C}$, $r > 0$ with $D_r(z) \subset \Omega$, and a function $\psi: D_r(0) \rightarrow \mathbb{C}$ with $\lim_{h \rightarrow 0} \psi(h) = 0$, so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation it is clear that f is continuous at z whenever f is holomorphic at z .

Example 1.1. It follows directly from the definition that the function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -1/z^2$. For all $0 \neq z \in \mathbb{C}$ we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{z+h} - \frac{1}{z} \right) = \lim_{h \rightarrow 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

Example 1.2. The function $f(z) = \bar{z}$ is not holomorphic. For any $z \in \mathbb{C}$ and $t \in \mathbb{R}$ we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1.$$

Proposition 1.7. Let $\Omega \subset \mathbb{C}$ be open and let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic at $z \in \Omega$. Then

- (1) $f + g$ is holomorphic at z with $(f + g)'(z) = f'(z) + g'(z)$.
- (2) fg is holomorphic at z with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Proof. We will only prove (2) because the proof of (1) is much simpler. Because f and g are holomorphic at z , they are also continuous there. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(fg)(z+h) - (fg)(z)}{h} &= \lim_{h \rightarrow 0} \left(\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right) \\ &= f'(z)g(z) + f(z)g'(z), \end{aligned}$$

which completes the proof. $\square$

Corollary 1.8. It's quite easy to prove that constant function of the form $f(z) = c$ for some $c \in \mathbb{C}$ and $f(z) = z$ are holomorphic. It follows immediately from Theorem 1.7 that all polynomials, functions of the form $p(z) = \sum_{k=0}^n a_k z^k$ are entire, with $p'(z) = \sum_{k=1}^n k a_k z^{k-1}$ for all $z \in \mathbb{C}$.

Proposition 1.9. A composition of holomorphic functions at z is holomorphic at z , with $(g \circ f)'(z) = g'(f(z))f'(z)$.

Corollary 1.10. Let $\Omega \subset \mathbb{C}$ be open and let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic at $z \in \Omega$. Suppose also that $g(z) \neq 0$. Then f/g is holomorphic at z with

$$(f/g)'(z) = \frac{f'(z)g(z) + f(z)g'(z)}{g(z)^2}.$$

Proof. Let $h: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be with $h(z) = 1/z$. We now have that

$$\begin{aligned} (f/g)'(z) &= (f \cdot (h \circ g))'(z) = f'(z)(h \circ g)(z) + f(z)(h \circ g)'(z) \\ &= f'(z)/g(z) + f(z)h'(g(z))g'(z) = f'(z)/g(z) - f(z)g(z)^{-2}g'(z). \end{aligned}$$

$\square$

Remark 1.13. Recall that $T: \mathbb{C} \rightarrow \mathbb{R}^2$ is the operator $T(x + yi) = (x, y)$.

Proposition 1.11. Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$, let $u, v: T(\Omega) \rightarrow \mathbb{R}$ be with $f(x + yi) = u(x, y) + iv(x, y)$ for $x + iy \in \Omega$, and let $F: T(\Omega) \rightarrow \mathbb{R}^2$ be with $F(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in T(\Omega)$. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p = (x_0, y_0)$, and suppose that f is holomorphic at z_0 . Then,

- (1) the partial derivatives of u and v exist at p , and

$$f'(z_0) = \frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p);$$

(2) The Cauchy–Riemann equations are satisfied:

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \quad \text{and} \quad \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

(3) F is differentiable at p with,

$$dF_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

Remark 1.14. Note that $u = \operatorname{Re} \circ f \circ T^{-1}$, $v = \operatorname{Im} \circ f \circ T^{-1}$ and $F = T \circ f \circ T^{-1}$. Thus, F is the map corresponding to f under the identification of $\mathbb{C}$ with $\mathbb{R}^2$ via T .

Remark 1.15. Note that from (3) we have

$$\det(dF_p) = \left(\frac{\partial u}{\partial x}(p) \right)^2 + \left(\frac{\partial v}{\partial x}(p) \right)^2.$$

From this and from (1), it follows that $\det(dF_p) > 0$ whenever $f'(z_0) \neq 0$. Moreover, we have that $\sqrt{\det(dF_p)} \cdot dF_p$ is an orthogonal matrix.

Remark 1.16. The Cauchy–Riemann equations are extremely useful. For example, if a function does not satisfy the Cauchy–Riemann equations at a point $z \in \mathbb{C}$, we know it is not holomorphic there.

Proof of Theorem 1.11.

(1) We can first let $t \rightarrow 0$ in $\mathbb{R}$ and see that

$$\begin{aligned} f'(z_0) &= \lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{h} \\ &= \lim_{t \rightarrow 0} \frac{u(x_0+t, y_0) - u(x_0, y_0) + iv(x_0+t, y_0) - iv(x_0, y_0)}{t} \\ &= \frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p). \end{aligned}$$

Similarly,

$$\begin{aligned} f'(z_0) &= \lim_{t \rightarrow 0} \frac{f(z+it) - f(z)}{h} \\ &= \lim_{t \rightarrow 0} \frac{u(x_0, y_0+t) - u(x_0, y_0) + iv(x_0, y_0+t) - iv(x_0, y_0)}{t} \\ &= \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p) \end{aligned}$$

which completes the proof of (1). From the equation

$$\frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p)$$

(2) This is an immediate arithmetic result of (1).

(3) Let $D: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator given in matrix form by

$$D := \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix}.$$

Since we have $F = T \circ f \circ T^{-1}$, we get that for $(h_1, h_2) = h \in \mathbb{R}^2$ with $p + h \in T(\Omega)$ that

$$\begin{aligned} \|F(p+h) - F(p) - D(h)\| &= \left\| T \left(f \circ T^{-1}(p+h) - f \circ T^{-1}(p) - T^{-1}D(h) \right) \right\| \\ &= \left| f(z_0 + (h_1 + ih_2)) - f(z_0) - T^{-1}D(h) \right|, \end{aligned}$$

and

$$\begin{aligned} T^{-1}D(h) &= \left(\frac{\partial u}{\partial x}(p)h_1 - \frac{\partial v}{\partial x}(p)h_2 \right) + i \left(\frac{\partial v}{\partial x}(p)h_1 + \frac{\partial u}{\partial x}(p)h_2 \right) \\ &= \left(\frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) \right) (h_1 + ih_2) = f'(z_0)(h_1 + ih_2), \end{aligned}$$

we get that

$$\begin{aligned} \frac{1}{\|h\|} \|F(p+h) - F(p) - D(h)\| \\ = \frac{1}{|h_1 + ih_2|} |f(z_0 + (h_1 + ih_2)) - f(z_0) - f'(z_0)(h_1 + ih_2)|. \end{aligned}$$

From this and since f is holomorphic at z_0 we get

$$\lim_{h \rightarrow 0} \frac{1}{\|h\|} \|F(p+h) - F(p) - D(h)\| = 0.$$

This means that F is differentiable at p with $dF_p = D$. The second equality in (3) follows from the Cauchy–Riemann equations, which completes the proof. $\square$

Remark 1.17. Unfortunately, satisfying the Cauchy–Riemann equations is not a sufficient condition to be holomorphic. For example the function

$$f(x + iy) = \sqrt{|x||y|}$$

satisfies them at the origin, yet it is not holomorphic at the origin. Luckily, we have the following proposition.

Proposition 1.12. *Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$, and let u and v be as in Theorem 1.11. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p := (x_0, y_0)$, and suppose that u and v are differentiable at p , so that $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$ and $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$. Then f is holomorphic at z_0 .*

Proof. Let $\epsilon > 0$ be such that $D_\epsilon(z_0) \subset \Omega$. Since u and v are differentiable at p , we have that for $(h_1, h_2) = h \in \mathbb{R}^2$ with $\|h\| < \epsilon$

$$\begin{aligned} u(p+h) - u(p) &= \frac{\partial u}{\partial x}(p)h_1 + \frac{\partial u}{\partial y}(p)h_2 + \|h\| \psi_1(h) \text{ for } \psi_1(h) \xrightarrow{h \rightarrow 0} 0 \\ v(p+h) - v(p) &= \frac{\partial v}{\partial x}(p)h_1 + \frac{\partial v}{\partial y}(p)h_2 + \|h\| \psi_2(h) \text{ for } \psi_2(h) \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

Thus, for $(h_1, h_2) = h \in D_\epsilon(0)$ we have (note that h is a complex number and not a vector in $\mathbb{R}^2$ now)

$$\begin{aligned} f(z_0 + h) - f(z_0) &= u(p + (h_1, h_2)) - u(p) + i(v(p + (h_1, h_2)) - v(p)) \\ &= \frac{\partial u}{\partial x}(p)h_1 + \frac{\partial u}{\partial y}(p)h_2 + i \left(\frac{\partial v}{\partial x}(p)h_1 + \frac{\partial v}{\partial y}(p)h_2 \right) \\ &\quad + |h| (\psi_1(h_1, h_2) + \psi_2(h_1, h_2)) \\ &= \left(\frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) \right) h + |h| (\psi_1(h_1, h_2) + \psi_2(h_1, h_2)). \end{aligned}$$

where the last equality follows from the Cauchy–Riemman equations. Since $\psi_1(h_1, h_2) + \psi_2(h_1, h_2) \rightarrow 0$ as $h_1 + ih_2 \rightarrow 0$, this completes the proof. $\square$

Remark 1.18. Satisfying the Cauchy–Riemman equations at a point $z_0 \in \mathbb{C}$ is equivalent to $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. Thus, a function $f = u + vi$ with continuous partial derivatives such that $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ it is also differentiable at z_0 .

1.3 Power series

Definition 1.26 (Power series). A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $\{a_n\}_{n \geq 0} \subset \mathbb{C}$. Given $z \in \mathbb{C}$, we say that the power series converges at z if the limit $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(z - z_0)^n$ exists in $\mathbb{C}$. If this limit does not exist, we say that the series diverges at z .

Definition 1.27 (Absolute convergence). Given a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, we say that it converges absolutely at $z \in \mathbb{C}$ if $\sum_{n=0}^{\infty} |a_n| \cdot |(z - z_0)|^n < \infty$.

Proposition 1.13. *If a power series converges absolutely at z then it also converges at z . This follows from the completeness of $\mathbb{C}$.*

Remark 1.19. In the following proposition we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proposition 1.14 (Hadamard's theorem). *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series and let $0 \leq R \leq \infty$ be given by*

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}.$$

Then for $z \in \mathbb{C}$ the series converges absolutely if $|z - z_0| < R$, and the series diverges if $|z - z_0| > R$.

Remark 1.20. The number R is called the radius of convergence of the power series, and the region $\{z \in \mathbb{C} : |z - z_0| < R\}$ is called the disc of convergence.

Proof of Theorem 1.14. Set $L := 1/R$. Suppose first that $0 < R \leq \infty$, so that $0 \leq L < \infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| < R$, then there exists $L < M < \infty$ so that $M|z - z_0| < 1$. By the definition of L (the limsup) there exists $N \geq 1$ so that $|a_n|^{\frac{1}{n}} < M$ for all $n > N$. Thus

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| \cdot |z - z_0|^n &= \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \left(|a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n \\ &\leq \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} (M|z - z_0|)^n < \infty. \end{aligned}$$

Suppose next that $0 \leq R < \infty$, so that $0 < L \leq \infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| > R$, then similarly there exists $0 < M < L$ so that $M|z - z_0| > 1$. Then, for every $N \geq 1$ there exists $n \geq N$ so that $|a_n|^{\frac{1}{n}} > M$. For such n we have

$$\begin{aligned} \left| \sum_{k=0}^n a_k(z - z_0)^k - \sum_{k=0}^{n-1} a_k(z - z_0)^k \right| &= |a_n| \cdot |z - z_0|^n \\ &= \left(|a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n > (M|z - z_0|)^n > 1, \end{aligned}$$

which shows that the partial sums do not form a Cauchy sequence. Thus the series diverges at z , which completes the proof. $\square$

Example 1.3. Consider the power series $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Because we have that for every $n \geq 1$

$$\sqrt[n]{(2n)!} \geq \sqrt[n]{n^n} = n,$$

we also have for every $n \geq 1$,

$$\left(\frac{1}{(2n)!}\right)^{\frac{1}{2n}} \leq \frac{1}{n^{\frac{1}{2}}} \quad \text{and} \quad \left(\frac{1}{(2n+1)!}\right)^{\frac{1}{2n+1}} \leq \frac{1}{n^{\frac{1}{2}}}.$$

Since $n^{-\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$ we get that the radius of convergence is ∞ for the series. The map $z \mapsto e^z$ is called the exponential function. We also have that

$$e^z e^w = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

Example 1.4. Consider the power series $f(z) := \sum_{n=0}^{\infty} z^n$. Since $1^{\frac{1}{n}} = 1$ we get that the radius of convergence in this case is 1. Thus f defines a function from $D_1(0)$ to $\mathbb{C}$. Moreover, since we have

$$(1-z) \sum_{n=0}^N z^n = 1 - z^{N+1},$$

we get for $z \in D_1(0)$ that

$$f(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N z^n = \lim_{N \rightarrow \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}.$$

Proposition 1.15. Let $f(z) := \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series, and let R be the radius of convergence of f . Then,

- (1) R is also the radius of convergence of $\sum_{n=0}^{\infty} n a_n(z - z_0)^{n-1}$;
- (2) suppose $R > 0$, then f is holomorphic in its disc of convergence with

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$

Proof.

- (1) Since $n^{1/n} \xrightarrow{n \rightarrow \infty} 1$ and $\frac{n-1}{n} \xrightarrow{n \rightarrow \infty} 1$, we have

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} |n a_n|^{\frac{1}{n-1}}$$

which gives the first part of the proposition.

- (2) By the chain rule, the derivative of $f(z)$ wouldn't change for any $z_0 \in \mathbb{C}$ so we may choose $z_0 = 0$ and then $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose $R > 0$, let $0 < r < R$, fix $w \in D_r(0)$, and define

$$g(z) := \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$

Since

$$\sum_{n=1}^{\infty} n |a_n| r^{n-1} \leq \sum_{n=1}^{\infty} n |a_n| |z|^{n-1} < \infty$$

there exists $N \geq 1$ so that

$$\sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \frac{\epsilon}{3}.$$

For $z \in D_r(0)$ set

$$S(z) = \sum_{n=0}^N a_n z^n \text{ and } E(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Since $w \in D_r(0)$ and S is holomorphic at w (since it is a polynomial), there exists $\delta > 0$ so that $D_\delta(w) \subset D_r(0)$ and

$$\begin{aligned} \left| \frac{f(w+h) - f(w)}{h} - g(w) \right| &\leq \left| \frac{S(w+h) - S(w)}{h} - S'(w) \right| + |S'(w) - g(w)| \\ &\quad + \left| \frac{E(w+h) - E(w)}{h} \right|. \end{aligned}$$

Recall that

$$S'(w) = \sum_{n=1}^N n a_n w^{n-1}.$$

Since $w \in B_r(0)$ we get

$$|S'(w) - g(w)| = \left| \sum_{n=N+1}^{\infty} n a_n w^{n-1} \right| \leq \left| \sum_{n=N+1}^{\infty} n |a_n| r^{n-1} \right| < \frac{\epsilon}{3}.$$

Notice that for each $n \geq 1$ and $a, b \in \mathbb{C}$,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = (a - b) \sum_{k=0}^{n-1} a^{n-k-1} b^k.$$

Using this fact, and since $w, w+h \in D_r(0)$,

$$|(w+h)^n - w^n| = \left| h \sum_{k=0}^{n-1} (w+h)^{n-k-1} w^k \right| \leq |h| n r^{n-1}.$$

We now have that

$$\begin{aligned} \left| \frac{E(w+h) - E(w)}{h} \right| &= \frac{1}{|h|} \left| \sum_{n=N+1}^{\infty} a_n ((w+h)^n - w^n) \right| \\ &= \frac{1}{|h|} \sum_{n=N+1}^{\infty} |a_n| \cdot |(w+h)^n - w^n| \\ &= \sum_{n=N+1}^{\infty} |a_n| n r^{n-1} < \frac{\epsilon}{3}. \end{aligned}$$

It now follows that

$$\left| \frac{f(w+h) - f(w)}{h} - g(w) \right| < \epsilon \text{ for } 0 \neq h \in D_\delta(0).$$

This shows that f is holomorphic at w with $f'(w) = g(w)$, which completes the proof.

□

Corollary 1.16. *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

Example 1.5. The function e^z is entire with

$$(e^z)' = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = e^z \text{ for all } z \in \mathbb{C}.$$

Example 1.6. The standard trigonometric functions are given by

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \text{ and } \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

It is easy to verify that in both cases the radius of convergence is ∞ . We also see that

$$(\cos z)' = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!} = -\sin z$$

and

$$(\sin z)' = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z.$$

From these equalities it is easy to check that $\sin z$ and $\cos z$ agree with their respective real versions. Moreover, a simple calculation gives the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

which are called Euler formulas. By adding these identities we get

$$e^{iz} = \cos z + i \sin z.$$

It follows that for all $x, y \in \mathbb{R}$ we have

$$e^{x+yi} = e^x e^{yi} = e^x (\cos y + i \sin y).$$

Definition 1.28 (Analytic function). Let $\Omega \subset \mathbb{C}$ be open. A function $f: \Omega \rightarrow \mathbb{C}$ is said to be analytic at $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, with radius of convergence $R > 0$, such that for some $0 < r < R$ with $D_r(z_0) \subset \Omega$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for } z \in D_r(z_0).$$

If f has a power series expansion at every point in Ω , we say that f is analytic on Ω .

Remark 1.21. From Theorem 1.15 we get that an analytic function on Ω is also holomorphic there. A deep theorem we shall prove later is that every holomorphic function is analytic. This is why the terms holomorphic and analytic are often used interchangeably.

1.4 Integration along paths

Definition 1.29 (Path). A path (in $\mathbb{C}$) is a continuous function $\gamma: [a, b] \rightarrow \mathbb{C}$, where $a, b \in \mathbb{R}$ with $a < b$.

Definition 1.30 (Closed path). A closed path is a path such that $\gamma(a) = \gamma(b)$.

Definition 1.31 (Simple path). A path is called simple if γ is injective. If the path is closed, we only require $\gamma|_{(a,b)}$ to be injective.

Remark 1.22. We write γ^* for the image of γ , i.e. $\gamma^* := \gamma([a, b])$. Note that γ^* is a compact subset of $\mathbb{C}$.

Definition 1.32 (Differentiable path). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path. We say that γ is differentiable at $t \in [a, b]$ if the following limit exists

$$\gamma'(t) = \lim_{h \rightarrow 0} \frac{\gamma(t+h) - \gamma(t)}{h},$$

where at the endpoints a, b the limit is one-sided.

Definition 1.33 (Smooth path). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path. We say that γ is smooth if $\gamma'(t)$ exists at all $t \in [a, b]$, and the map $\gamma': [a, b] \rightarrow \mathbb{C}$ is continuous.

Definition 1.34 (Piecewise-smooth path). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path. We say that γ is piecewise-smooth if there exist points $a = a_0 < a_1 < \dots < a_n = b$ so that for each $0 \leq k < n$ the restriction of γ to $[a_k, a_{k+1}]$ is smooth.

Remark 1.23. Note that a piecewise-smooth path is continuous and not just piecewise continuous. Also, from now on every time we say ‘path’ we mean ‘piecewise-smooth path’.

Definition 1.35 (Integral). Given a continuous function $f: [a, b] \rightarrow \mathbb{C}$ we define

$$\int_a^b f(t)dt := \int_a^b \operatorname{Re}(f(t))dt + i \int_a^b \operatorname{Im}(f(t))dt.$$

Definition 1.36 (Integral along a path). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path, and let $a = a_0 < a_1 < \dots < a_n = b$ be such that γ is smooth on $[a_k, a_{k+1}]$ for each $0 \leq k < n$. Given a continuous $f: \gamma^* \rightarrow \mathbb{C}$, we define the integral of f along γ as follows

$$\int_{\gamma} f(z)dz := \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(\gamma(t))\gamma'(t)dt.$$

Remark 1.24. Notice that the integrand $f(\gamma(t))\gamma'(t)$ is well defined and continuous at each $t \in [a, b] \setminus \{a_1, \dots, a_{n-1}\}$. Thus, we can also write

$$\int_{\gamma} f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Definition 1.37 (Length of a path). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path, and let $a = a_0 < a_1 < \dots < a_n = b$ be such that γ is smooth on $[a_k, a_{k+1}]$ for each $0 \leq k < n$. The length of γ is defined as follows,

$$\operatorname{length}(\gamma) := \int_a^b |\gamma'(t)|dt.$$

Remark 1.25. The intuition is that we sum infinitely many terms of the form $|\gamma(t + \Delta t) - \gamma(t)|$, and from Lagrange’s theorem this is equal to $|\gamma'(t_{\text{mid}})| \cdot \Delta t$ for some $t < t_{\text{mid}} < t + \Delta t$. Thus the integral of $|\gamma'(t)|$ over $[a, b]$ is equal the length of the path.

Example 1.7 (Length of a circle). Let $z_0 \in \mathbb{C}$ and $r > 0$ be given. Let $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ be with $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$. It is clear that γ is a smooth, closed and simple path. Let $f: C_r(z_0) \rightarrow \mathbb{C}$ be a continuous. Using a simple substitution we see that

$$\int_{\gamma} f(z)dz = \int_{C_r(z_0)} f(z)dz = \int_0^{2\pi} f(z_0 + re^{i\theta})ire^{i\theta}d\theta.$$

Also,

$$\text{length}(\gamma) = \text{length}(C_r(z_0)) = \int_0^{2\pi} |ire^{i\theta}|d\theta = \int_0^{2\pi} |r|d\theta = 2\pi r.$$

Remark 1.26. The curve γ in this example is called a positively oriented circle with center z_0 and radius r . That is because when travelling along the curve, the interior of the circle would be on the left, and the exterior on the right. If we chose $\gamma(\theta) = z_0 + re^{-i\theta}$ then it would be the other way around, so we would call it a negatively oriented circle. Every simple closed curve has such an orientation by a heavy theorem called the Jordan curve theorem which we will not prove in these notes.

Example 1.8. Let $\alpha, \beta \in \mathbb{C}$ be given. Let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be such that $\gamma(t) = t\beta + (1-t)\alpha$ for $t \in [0, 1]$. Then γ is a smooth path, and it is simple if and only if $\alpha \neq \beta$. We denote the image of γ by $[\alpha, \beta]$. Let $f: [\alpha, \beta] \rightarrow \mathbb{C}$. We have

$$\int_{[\alpha, \beta]} f(z)dz = \int_0^1 f(t\beta + (1-t)\alpha)(\beta - \alpha)dt.$$

Also,

$$\text{length}(\gamma) = \int_0^1 |\beta - \alpha|dt = |\beta - \alpha|.$$

Example 1.9. Let $\gamma: [\alpha, \beta] \rightarrow \mathbb{C}$ be a path, and let $\gamma^-: [\alpha, \beta]$ be such that $\gamma^-(t) = \gamma(\alpha + \beta - t)$. It is clear that γ^- is also a path. We call γ^- the path opposite to γ . Given a continuous $f: \gamma^* \rightarrow \mathbb{C}$,

$$\int_{\gamma^-} f(z)dz = \int_a^b f(\gamma^-(t))(\gamma^-)'(t)dt = - \int_a^b f(\gamma(a + b - t))\gamma'(b + a - t)dt.$$

Thus by the substitution $s = b + a - t$,

$$\int_{\gamma^-} f(z)dz = - \int_a^b f(\gamma(s))\gamma'(s)dt = - \int_{\gamma} f(z)dz.$$

Moreover, by a similar substitution,

$$\text{length}(\gamma^-) = \int_a^b |\gamma'(b + a - t)|dt = \int_a^b |\gamma'(s)|dt = \text{length}(\gamma).$$

Remark 1.27. It is worth pointing out that if $w, \eta \in \mathbb{C}$ and γ is the oriented interval from w to η , then γ^- is the oriented interval from η to w .

Proposition 1.17. For continuous $f, g: [a, b] \rightarrow \mathbb{C}$ and $\alpha \in \mathbb{C}$,

$$\alpha \int_a^b f(t)dt = \int_a^b \alpha f(t)dt \text{ and } \int_a^b (f + g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$

Moreover,

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt.$$

Proof. The first two parts of the proposition are rather simple so we will only prove the final part. We may assume that $\int_a^b f(t)dt \neq 0$. Set $\theta := \arg(\int_a^b f(t)dt)$. Then,

$$\left| \int_a^b f(t)dt \right| = e^{-i\theta} \int_a^b f(t)dt = \int_a^b e^{-i\theta} f(t)dt.$$

Since the last expression is real we can use properties of the Riemann integral and the fact that $|e^{-i\theta}| = 1$ to get

$$\left| \int_a^b f(t)dt \right| = \int_a^b \operatorname{Re} \left(e^{-i\theta} f(t) \right) dt \leq \int_a^b |e^{-i\theta} f(t)| dt = \int_a^b |f(t)| dt,$$

which completes the proof of the proposition. $\square$

Proposition 1.18. *Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path. Then,*

(1) *for every continuous $f, g: \gamma^* \rightarrow \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$*

$$\int_{\gamma} \alpha f(z) + \beta g(z) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz.$$

(2) *for every continuous $f: \gamma^* \rightarrow \mathbb{C}$*

$$\left| \int_{\gamma} f(z) dz \right| \leq \|f\|_{\gamma^*} \cdot \operatorname{length}(\gamma).$$

Remark 1.28. Part (2) of the proposition is sometimes called the estimation lemma, or the *ML-inequality*.

Proof of Theorem 1.18. The first property follows from the previous proposition, and its proof is omitted. Given a continuous $f: \gamma^* \rightarrow \mathbb{C}$

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \leq \|f\|_{\gamma^*} \int_a^b |\gamma'(t)| dt = \|f\|_{\gamma^*} \cdot \operatorname{length}(\gamma),$$

which completes the proof. $\square$

Definition 1.38 (Equivalence of paths). Two paths $\gamma_1: [a, b] \rightarrow \mathbb{C}$ and $\gamma_2: [c, d] \rightarrow \mathbb{C}$ are said to be equivalent if there exists a continuously differentiable bijection $\varphi: [a, b] \rightarrow [c, d]$ so that $\varphi'(t) > 0$ and $\gamma_1(t) = \gamma_2(\varphi(t))$ for all $t \in [a, b]$.

Remark 1.29. Intuitively, this means that the paths are exactly identical except for the “speed” we travel them. Since we don’t want to consider almost identical paths that only differ in their direction identical, we add the requirement $\varphi'(t) > 0$.

Proposition 1.19. *Let $\gamma_1: [a, b] \rightarrow \mathbb{C}$ and $\gamma_2: [c, d] \rightarrow \mathbb{C}$ be equivalent paths. Then $\operatorname{length}(\gamma_1) = \operatorname{length}(\gamma_2)$, and $\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$ for every continuous function $f: \gamma_1^* \rightarrow \mathbb{C}$.*

Proof. Since γ_1 and γ_2 are equivalent, there exists a continuously differentiable bijection φ from $[a, b]$ to $[c, d]$ so that $\varphi'(t) > 0$ and $\gamma_1(t) = \gamma_2(\varphi(t))$ for all $t \in [a, b]$. Thus,

$$\operatorname{length}(\gamma_1) = \int_a^b |\gamma_1'(t)| dt = \int_a^b |\gamma_2'(\varphi(t)) \varphi'(t)| dt = \int_a^b |\gamma_2'(s)| ds = \operatorname{length}(\gamma_2).$$

Similarly,

$$\begin{aligned}\int_{\gamma_1} f(z)dz &= \int_a^b f(\gamma_1(t))\gamma_1'(t)dt = \int_a^b f(\gamma_2(\varphi(t)))\gamma_2'(\varphi(t))\varphi'(t)dt \\ &= \int_c^d f(\gamma_2(s))\gamma_2'(s)ds = \int_{\gamma_2} f(z)dz,\end{aligned}$$

which completes the proof. $\square$

Definition 1.39 (Primitive function). Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f: \Omega \rightarrow \mathbb{C}$. A primitive for f on Ω is a function $F: \Omega \rightarrow \mathbb{C}$ that is holomorphic on Ω with $F'(z) = f(z)$ for all $z \in \Omega$.

Remark 1.30. The following are the complex version of the chain rule and fundamental theorem of calculus.

Lemma 1.20. Let $\emptyset \neq \Omega \subset \mathbb{C}$ be open, let $F: \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω , and let $\gamma: [a, b] \rightarrow \Omega$ be a smooth path. Then $F \circ \gamma$ is also a smooth path with $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$ for all $t \in [a, b]$.

Proposition 1.21. Let $\emptyset \neq \Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be continuous, let $F: \Omega \rightarrow \mathbb{C}$ be a primitive for f on Ω , and let $\gamma: [a, b] \rightarrow \Omega$ be a path. Then,

$$\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Let $a = a_0 < a_1 < \dots < a_n = b$ be such that γ is piecewise-smooth on $[a_k, a_{k+1}]$ for all $0 \leq k < n$. We then have that

$$\begin{aligned}\int_{\gamma} f(z)dz &= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(\gamma(t))\gamma'(t)dt = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} F'(\gamma(t))\gamma'(t)dt \\ &= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} (F \circ \gamma)'(t)dt = \sum_{k=0}^{n-1} (F(\gamma(a_{k+1})) - F(\gamma(a_k)))\end{aligned}$$

where the last equality follows from the fundamental theorem of calculus. Since the last sum is telescopic we have that

$$\int_{\gamma} f(z)dz = F(\gamma(a_n)) - F(\gamma(a_0)) = F(\gamma(b)) - F(\gamma(a)),$$

which completes the proof. $\square$

Corollary 1.22. Let $\emptyset \neq \Omega \subset \mathbb{C}$ be open, let $\gamma: [a, b] \rightarrow \Omega$ be a closed path, and let $f: \Omega \rightarrow \mathbb{C}$ be continuous. Suppose that f has a primitive on Ω , then $\int_{\gamma} f(z)dz = 0$.

Proof. Since γ is closed we have $\gamma(a) = \gamma(b)$. This together with the previous proposition completes the proof. $\square$

Corollary 1.23. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a closed path. Then $\int_{\gamma} p(z)dz = 0$ for every polynomial $p: \mathbb{C} \rightarrow \mathbb{C}$.

Remark 1.31. This corollary follows immediately from the fact that $z \mapsto z^{n+1}/(n+1)$ is a primitive for $z \mapsto z^n$ and previous propositions. However, Theorem 1.22 can also shine when trying to prove a function has no primitive.

Example 1.10. Let $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be defined with $f(z) = 1/z$. We have that

$$\int_{C_1(0)} f(z) dz = \int_0^{2\pi} f(e^{i\theta}) i e^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i.$$

From Theorem 1.22 we get that f does not have a primitive on $\mathbb{C} \setminus \{0\}$.

Corollary 1.24. Let $\Omega \subset \mathbb{C}$ be a region, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Suppose that $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

Proof. We first need the following lemma:

Lemma 1.25. Let $\Omega \subset \mathbb{C}$ be a region. Then for every $z, w \in \Omega$ there exists a path $\gamma: [0, 1] \rightarrow \Omega$ with $\gamma(0) = z$ and $\gamma(1) = w$.

This lemma means that a region is also path connected. Since this lemma is really a topological result, we will omit its proof. Let $z, w \in \Omega$ be given. Then there exists a path $\gamma: [0, 1] \rightarrow \Omega$ with $\gamma(0) = z$ and $\gamma(1) = w$. Now since $f'(z) = 0$ on Ω , from Theorem 1.22 we can conclude that

$$0 = \int_{\gamma} f'(z) dz = f(\gamma(1)) - f(\gamma(0)) = f(w) - f(z) \implies f(w) = f(z).$$

This implies that f is constant and completes the proof. $\square$

1.5 The index of a point with respect to a path

Lemma 1.26. Let Ω be a nonempty open subset of $\mathbb{C}$. Then there exists a finite or countable collection $\{\Omega_j\}_{j \in J}$ of open subsets of Ω so that

- (1) Ω_j is connected for each $j \in J$.
- (2) $\Omega_{j_1} \cap \Omega_{j_2} = \emptyset$ for distinct $j_1, j_2 \in J$.
- (3) $\{\Omega_j\}_{j \in J}$ is an open cover of Ω .

These properties do not only uniquely determine $\{\Omega_j\}_{j \in J}$, but also if $\mathbb{C} \setminus \Omega$ is bounded then there exists a unique $j \in J$ so that Ω_j is unbounded.

Remark 1.32. The sets Ω_j are called the connected components of Ω . These sets are path-connected as well.

Theorem 1.27 (Index of a point). Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a closed path, set $\Omega := \mathbb{C} \setminus \gamma^*$, and for $z \in \Omega$ define

$$\text{Ind}_{\gamma}(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}.$$

Then,

- (1) Ind_{γ} is an integer valued function on Ω ;
- (2) Ind_{γ} is constant on each connected component of Ω ;
- (3) $\text{Ind}_{\gamma} = 0$ on the unique unbounded connected component of Ω .

Remark 1.33. Since γ^* is compact, Ω is open and it has a unique unbounded connected component.

Proof. For $z \in \Omega$ we have from definition of integral over a curve that

$$\text{Ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{dw}{w-z} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)-z} dt.$$

Recall from Example 1.6 that for $\eta = x + iy \in \mathbb{C}$ we get $e^\eta = e^x(\cos(y) + i\sin(y))$. Thus $e^\eta = 1$ if and only if $x = 0$ and $y \in (2\pi i)\mathbb{Z}$ which happens if and only if $\eta/(2\pi i) \in \mathbb{Z}$. Thus $\text{Ind}_\gamma(z) \in \mathbb{Z}$ if and only if $\varphi(b) = 1$ for the function $\varphi: [a, b] \rightarrow \mathbb{C}$ defined by

$$\varphi(s) := \exp\left(\frac{\gamma'(t)}{\gamma(t)-z} dt\right).$$

Let $S \subset [a, b]$ be the finite set at which γ is not differentiable. Then by the chain rule, and the complex version of the fundamental theorem of calculus we get that for $s \in [a, b] \setminus S$

$$\varphi'(s) = \frac{\gamma'(s)}{\gamma(s)-z} \exp\left(\frac{\gamma'(t)}{\gamma(t)-z} dt\right) = \frac{\gamma'(s)}{\gamma(s)-z} \varphi(s).$$

Thus, for $s \in [a, b] \setminus S$,

$$\left(\frac{\varphi}{\gamma-z}\right)'(s) = \frac{\varphi'(s)(\gamma(s)-z) - \varphi(s)\gamma'(s)}{(\gamma(s)-z)^2} = 0.$$

Since $\frac{\varphi}{\gamma-z}$ is continuous and S is finite, it follows that $\frac{\varphi}{\gamma-z}$ is constant on $[a, b]$. Thus,

$$\frac{\varphi(b)}{\gamma(b)-z} = \frac{\varphi(a)}{\gamma(a)-z} = \frac{1}{\gamma(a)-z}.$$

From this, and since γ is closed, we obtain that $\varphi(b) = 1$, so as we have noted before this implies that $\text{Ind}_\gamma(z) \in \mathbb{Z}$.

Next we show that $\text{Ind}_\gamma(z)$ is a continuous function. Let $a \leq c < d \leq b$ be such that $\gamma|_{[a,b]}$ is smooth. Let $z_0 \in \Omega$ and $\epsilon > 0$ be given. Since $[c, d]$ is compact, there exists $\delta > 0$ so that $D_\delta(z_0) \subset \Omega$ and

$$\left| \frac{\gamma'(t)}{\gamma(t)-z} - \frac{\gamma'(t)}{\gamma(t)-z_0} \right| < \epsilon$$

for all $(t, z) \in [c, d] \times D_\delta(z_0)$ (we used compactness of $[c, d]$ to say that there exists one δ that works *uniformly* for all $(t, z) \in [c, d] \times D_\delta(z_0)$). Thus, for $z \in D_\delta(z_0)$ we get

$$\left| \int_c^d \frac{\gamma'(t)}{\gamma(t)-z} dt - \int_c^d \frac{\gamma'(t)}{\gamma(t)-z_0} dt \right| < \epsilon|d-c|,$$

which shows that $z \mapsto \int_c^d \frac{\gamma'(t)}{\gamma(t)-z} dt$ is continuous. Since γ is piecewise-smooth, it follows that Ind_γ is continuous. Since Ind_γ is an integer valued function, it follows that Ind_γ is constant on each connected component of Ω .

Finally, let Ω_0 be the unique unbounded component of Ω . By the estimation lemma it follows that

$$|\text{Ind}_\gamma(z)| \leq \sup_{w \in \gamma^*} \frac{1}{|w-z|} \cdot \text{length}(\gamma).$$

Since γ^* is compact and Ω_0 is unbounded, there exists $z_0 \in \Omega_0$ so that

$$\sup_{w \in \gamma^*} \frac{1}{|w-z_0|} \cdot \text{length}(\gamma) < 1.$$

Since $\text{Ind}_\gamma(z_0) \in \mathbb{Z}$ it follows that $\text{Ind}_\gamma(z_0) = 0$, and since Ind_γ is constant on each connected component of Ω , we get $\text{Ind}_\gamma(z) = 0$ for all $z \in \Omega_0$, which completes the proof. $\square$

Definition 1.40 (Index of a point). The number

$$\text{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}$$

is called the index of z with respect to γ . Intuitively the index of z is the number of times γ winds around the point $z \in \Omega$.

Proposition 1.28. *Let $z_0 \in \mathbb{C}$ and $r > 0$ be given, and let $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ be the positively oriented circle with center at z_0 and radius r . Then $\text{Ind}_\gamma(z) = 1$ for $z \in D_r(z_0)$ and $\text{Ind}_\gamma(z) = 0$ for $z \in \mathbb{C} \setminus \overline{D}_r(z_0)$. Recall that $D_r(z_0)$ is an open disc around z_0 and it does not include its boundary.*

Proof. The set $\Omega := \mathbb{C} \setminus \gamma^*$ has two connected components. Since $\mathbb{C} \setminus \overline{D}_r(z_0)$ is unbounded we have from Theorem 1.27 part (3) that $\text{Ind}_\gamma(z) = 0$ for $z \in \mathbb{C} \setminus \overline{D}_r(z_0)$. We also have that the index at the center of the circle is

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{i\theta}}{(z_0 + re^{i\theta}) - z_0} dt = 1.$$

From this, and since Ind_γ is constant on $D_r(z_0)$, it follows that $\text{Ind}_\gamma(z) = 1$ for all $z \in D_r(z_0)$ which completes the proof. $\square$

2 Cauchy's theorem and its applications

2.1 Cauchy's theorem for a triangle

Definition 2.1 (Triangle notation). Let (a, b, c) be an ordered triple of complex number. We denote $\Delta(a, b, c) \subset \mathbb{C}$ the convex hull of $\{a, b, c\}$. Intuitively, $\Delta(a, b, c)$ is triangle with vertices (a, b, c) in $\mathbb{C}$.

Definition 2.2 (Integration over a triangle). For a continuous $f: \partial\Delta \rightarrow \mathbb{C}$, we define

$$\int_{\partial\Delta} f(z)dz := \int_{[a,b]} f(z)dz + \int_{[b,c]} f(z)dz + \int_{[c,a]} f(z)dz.$$

This is equivalent to defining $\int_{\partial\Delta} f(z)dz$ as the integral of f over the simple closed path γ that travels along the triangle.

Remark 2.1. We sometimes write $\text{length}(\partial\Delta)$ instead of $\text{length}(\gamma)$, and we get

$$\text{length}(\partial\Delta) = |a - b| + |b - c| + |c - a|.$$

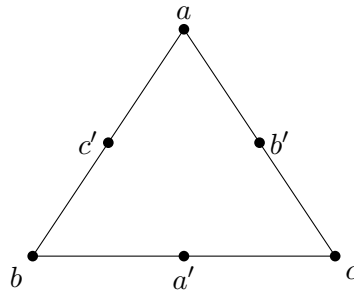
Theorem 2.1 (Cauchy's theorem for a triangle). *Let $\Omega \subset \mathbb{C}$ be open, let Δ be a triangle in Ω , let $p \in \Omega$, and let $f: \Omega \rightarrow \mathbb{C}$ be continuous on Ω and holomorphic on $\Omega \setminus \{p\}$. Then $\int_{\partial\Delta} f(z)dz = 0$.*

Remark 2.2. Under the assumptions of the theorem we also get that f is holomorphic at p . However, assuming it is only holomorphic on $\Omega \setminus \{p\}$ is useful for the proof of Cauchy's formula.

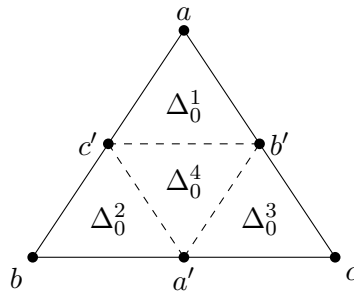
Proof. Let $a, b, c \in \mathbb{C}$ be with $\Delta = \Delta(a, b, c)$. Set

$$\begin{aligned} J &:= \int_{\partial\Delta} f(z)dz \\ \Delta_0 &:= \Delta \\ L &:= \text{length}(\partial\Delta_0) \\ D &:= \text{diam}(\Delta_0) \end{aligned}$$

and assume that $p \notin \Delta$. Let a', b' and c' be the midpoints of $[b, c]$, $[c, a]$ and $[a, b]$ respectively.



Set $\Delta_0^1 := \Delta(a, c', b')$, $\Delta_0^2 := \Delta(c', b, a')$, $\Delta_0^3 := \Delta(a', c, b')$, $\Delta_0^4 := \Delta(a', b', c')$.



We now have that

$$J = \sum_{j=1}^4 \int_{\partial \Delta_0^j} f(z) dz.$$

Thus, by the triangle inequality there exists $1 \leq i \leq 4$ so that

$$|J| \leq 4 \left| \int_{\partial \Delta_0^i} f(z) dz \right|.$$

We can now set $\Delta_1 := \Delta_0^i$ and repeat the preceding argument with Δ_1 instead of Δ_0 , which generates a sequence of closed triangles $\{\Delta_n\}_{n \geq 0}$ so that for each $n \geq 0$ we have that

$$\Delta_{n+1} \subset \Delta_n \quad \text{and} \quad \text{diam}(\Delta_n) = 2^{-n}D \quad \text{and} \quad \text{length}(\partial \Delta_n) = 2^{-n}L$$

and also

$$|J| \leq 4^n \left| \int_{\partial \Delta_n} f(z) dz \right|.$$

By Cantor's intersection lemma, there exists a unique $z_0 \in \mathbb{C}$ so that $\cap_{n \geq 0} \Delta_n = \{z_0\}$. Now since $z_0 \in \Delta_0 \subset \Omega \setminus \{p\}$, the function f is holomorphic at z_0 , so for every $\epsilon > 0$ there exists $r > 0$ so that $D_r(z_0) \subset \Omega$ and

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon |z - z_0|$$

for all $z \in D_r(z_0)$. Since $\text{diam}(\Delta_n) \rightarrow 0$ as $n \rightarrow \infty$ there exists $n \geq 1$ so that $\Delta_n \subset D_r(z_0)$. Then for every $z \in \Delta_n$ we have

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon |z - z_0| \leq \epsilon \cdot \text{diam}(\Delta_n) = \epsilon 2^{-n}D.$$

Since $-f(z_0) - (z - z_0)f'(z_0)$ is a polynomial from Theorem 1.23 we can write that

$$\int_{\partial \Delta_n} f(z) dz = \int_{\partial \Delta_n} f(z) - f(z_0) - (z - z_0)f'(z_0) dz.$$

From the estimation lemma (Theorem 1.18) we now have that

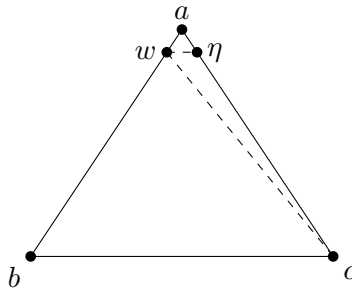
$$\left| \int_{\partial \Delta_n} f(z) dz \right| \leq \epsilon 2^{-n}D \cdot \text{length}(\partial \Delta_n) = \epsilon 4^{-n}DL.$$

Using the inequality from earlier

$$|J| \leq 4^n \left| \int_{\partial \Delta_n} f(z) dz \right| \implies |J| \leq \epsilon DL.$$

Since this inequality holds for any $\epsilon > 0$ this implies that $J = 0$ which completes the proof for the case $p \notin \Delta$.

Next suppose that $p \in \{a, b, c\}$. Without loss of generality assume that $p = a$. We then let $0 < t < 1$ be some small value and set $w := tb + (1 - t)a$ and $\eta := tc + (1 - t)a$ as can be seen in the following diagram.



We now have that

$$J = \int_{\partial\Delta(a,w,\eta)} f(z)dz + \int_{\partial\Delta(w,b,c)} f(z)dz + \int_{\partial\Delta(w,c,\eta)} f(z)dz.$$

Moreover, since $p \notin \Delta(w,b,c)$ and $p \notin \Delta(w,c,\eta)$ by the previous result

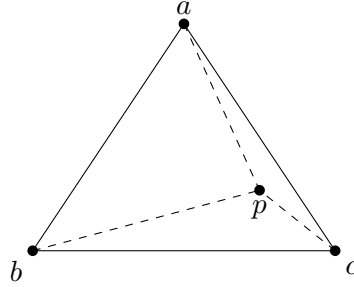
$$\int_{\partial\Delta(w,b,c)} f(z)dz = \int_{\partial\Delta(w,c,\eta)} f(z)dz = 0.$$

Thus we finally have by the estimation lemma

$$|J| = \left| \int_{\partial\Delta(a,w,\eta)} f(z)dz \right| \leq \|f\|_{\Delta} \cdot \text{length}(\partial\Delta(a,w,\eta)).$$

Since f is continuous on $\Delta(a,b,c)$ the value of $\|f\|_{\Delta}$ is finite, and since this holds for any $t > 0$ and $\text{length}(\partial\Delta(a,w,\eta)) \rightarrow 0$ as $t \rightarrow 0$ we get that $J = 0$ in this case as well.

Finally, assume that $p \in \Delta \setminus \{a,b,c\}$. We then have a triangle of the form



and since by the previous case

$$0 = \int_{\partial\Delta(a,b,p)} f(z)dz = \int_{\partial\Delta(b,c,p)} f(z)dz = \int_{\partial\Delta(c,a,p)} f(z)dz,$$

and since J is the sum of these three integrals, we get that $J = 0$ which completes the proof of this theorem. $\square$

2.2 Cauchy's theorem in a convex set

Theorem 2.2 (Cauchy's theorem in a convex set). *Let $\Omega \subset \mathbb{C}$ be convex and open, let $p \in \Omega$, and let $f: \Omega \rightarrow \mathbb{C}$. Suppose that f is continuous on Ω and holomorphic on $\Omega \setminus \{p\}$. Then f has a primitive on Ω , and $\int_{\gamma} f(z)dz = 0$ for every closed path $\gamma: [a,b] \rightarrow \Omega$.*

Proof. By Theorem 1.22 it suffices to show that f has a primitive on Ω . Fix $z_0 \in \Omega$, and define $F: \Omega \rightarrow \mathbb{C}$ as such

$$F(z) := \int_{[z_0,z]} f(z)dz.$$

Since Ω is convex this function is well define. We will show that F is a primitive for f on Ω . Let $z_1 \in \Omega$ and $\epsilon > 0$ be given. Since f is continuous at z_1 there exists $\delta > 0$ such that $|f(z) - f(z_1)| < \epsilon$ for all $z \in D_{\delta}(z_1)$. Let $0 \neq h \in D_{\delta}(0)$, we then have that

$$\begin{aligned} F(z_1 + h) - F(z_1) &= \int_{[z_0, z_1+h]} f(z)dz + \int_{[z_0, z_1+h]} f(z)dz \\ &= \int_{\partial\Delta(z_0, z_1+h, z_1)} f(z)dz + \int_{[z_1+h, z_1]} f(z)dz \\ &= \int_{[z_1, z_1+h]} f(z)dz. \end{aligned}$$

where the last equality follows from Theorem 2.1. Moreover, from Example 1.8 we get that

$$\frac{1}{h} \int_{[z_1, z_1+h]} f(z_1) dz = \frac{1}{h} f(z_1) \int_0^1 h dt = f(z_1).$$

Now since $[z_1, z_1 + h] \subset D_\delta(z_1)$ we get

$$\begin{aligned} \left| \frac{F(z_1 + h) - F(z_1)}{h} - f(z_1) \right| &= \left| \frac{1}{h} \int_{[z_1, z_1+h]} f(z) - f(z_1) dz \right| \\ &< |h|^{-1} \text{length}([z_1, z_1 + h]) \epsilon = \epsilon. \end{aligned}$$

This shows that F is holomorphic at z_1 with $F'(z_1) = f(z_1)$, which completes the proof of this theorem. $\square$

Remark 2.3. Here are some applications for Cauchy's theorem in a convex set.

Example 2.1. We will show that for $\xi \in \mathbb{R}$ we have

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx.$$

This means that $x \mapsto e^{-\pi x^2}$ is its own Fourier transform.

Recall that by using polar coordinates we get

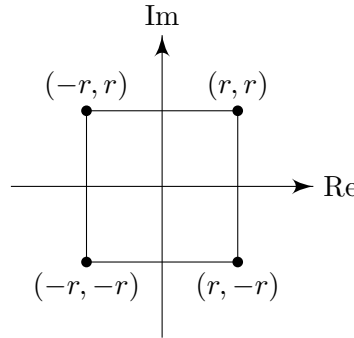
$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \pi.$$

This gives that

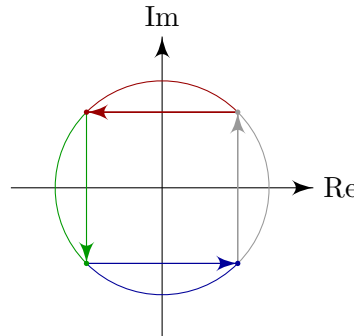
$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1,$$

which shows the result for $\xi = 0$.

Example 2.2 (Index in squares). Let S_r denote the closed square with edges of size $2r$ centered around the origin. The following is a diagram of its circumference.



Then let r , g , b and w be the closed red, green, blue and white paths in the diagram below. Each path starts from a vertex of the square then goes over a quarter of $C_{\sqrt{2}r}$ then over the natural edge that closes the path.



Denote γ the closed path that goes over the circumference of the square. We now have that

$$\int_r \frac{dz}{z - z_0} + \int_g \frac{dz}{z - z_0} + \int_b \frac{dz}{z - z_0} + \int_w \frac{dz}{z - z_0} = \int_\gamma \frac{dz}{z - z_0} - \int_{C_{\sqrt{2}r}} \frac{dz}{z - z_0}.$$

However, from Theorem 2.2 we get that all the integral on the left hand side are equal to zero, which means that

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_{C_{\sqrt{2}r}} \frac{dz}{z - z_0}.$$

This means that $\text{Ind}_\gamma(z_0) = 1$ if and only if $z_0 \in \text{Int}(S_r)$ and $\text{Ind}_\gamma(z_0) = 0$ when $z_0 \in \mathbb{C} \setminus S_r$.

2.3 Cauchy's formula in a convex set and its applications

Theorem 2.3 (Cauchy's formula in a convex set). *Let $\Omega \subset \mathbb{C}$ be convex and open, let $\gamma: [a, b] \rightarrow \Omega$ be a closed path, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then for all $z_0 \in \Omega \setminus \gamma^*$ we have*

$$f(z_0) \cdot \text{Ind}(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. Fix $z_0 \in \Omega \setminus \gamma^*$. Let $g: \Omega \rightarrow \mathbb{C}$ be with

$$g(z) := \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & z \in \Omega \setminus z_0 \\ f'(z_0), & z = z_0 \end{cases}.$$

Since f is holomorphic on Ω , the function g is continuous on Ω and holomorphic on $\Omega \setminus \{z_0\}$. Thus by Cauchy's theorem on a convex set we have

$$0 = \frac{1}{2\pi i} \int_\gamma g(z) dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz.$$

This implies that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = f(z_0) \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz = f(z_0) \cdot \text{Ind}_\gamma(z_0)$$

where the last inequality follows from the definition of $\text{Ind}_\gamma(z_0)$. □

Remark 2.4. Theorem 2.3 tells us that the value of $f(z_0)$ is determined completely by the values of $f(z)$ on any closed curve around z_0 .

Corollary 2.4 (Cauchy's formula). *Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \Omega$ and $r > 0$ be with $\overline{D_r}(z_0) \subset \Omega$. Then,*

$$f(w) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - w} dz$$

for all $w \in D_r(z_0)$.

Proof. Since Ω is open and $\overline{C_r}(z_0)$ is compact, there exists $r' > r$ such that $D_{r'}(z_0) \subset \Omega$. Fix $w \in D_r(z_0)$ and let γ be the positively oriented circle with center z_0 and radius r . Since $D_{r'}(z_0)$ is open and convex and f is holomorphic on $D_{r'}(z_0)$ then by Cauchy's formula we get

$$f(w) \cdot \text{Ind}(w) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - w} dz.$$

However, since $w \in C_r(z_0)$ we get from an earlier result that $\text{Ind}(w) = 1$ and thus

$$f(w) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - w} dz$$

which completes the proof. □

Lemma 2.5. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a path, let $f_1, f_2, \dots: \gamma^* \rightarrow \mathbb{C}$ be continuous functions and suppose that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on γ^* (i.e. $\|f - f_n\|_{\gamma^*} \rightarrow 0$ as $n \rightarrow \infty$). Then $\int_{\gamma} f_n dz \xrightarrow{n \rightarrow \infty} \int_{\gamma} f dz$.

Proof.

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int_{\gamma} f(z) dz - \int_{\gamma} f_n(z) dz \right| &= \limsup_{n \rightarrow \infty} \left| \int_{\gamma} (f_n - f)(z) dz \right| \\ &\leq \limsup_{n \rightarrow \infty} \|f_n - f\|_{\gamma^*} \cdot \text{length}(\gamma) = 0. \end{aligned}$$

□

Theorem 2.6. Let $\Omega \subset \mathbb{C}$ be open and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then f has infinitely many complex derivatives in Ω . Moreover, given $z_0 \in \Omega$ and $r > 0$ with $\overline{D}_r(z_0) \subset \Omega$, for all $n \geq 0$ we have

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z - w)^{n+1}} dz$$

for all $w \in D_r(z_0)$, where $f^{(n)}$ denotes the n th complex derivative of f .

Proof. Let $z_0 \in \Omega$ and $r > 0$ be with $\overline{D}_r(z_0) \subset \Omega$. The proof will be carried out using induction on n . For $n = 0$ the equality follows directly from Theorem 2.4.

Let $n \geq 1$ and suppose that the equality holds for $n - 1$. Fix $w \in D_r(z_0)$ and let $\delta > 0$ be with $B_{\delta}(w) \subset D_r(z_0)$. For $h \in D_{\delta}(0)$ we get

$$\frac{f^{(n-1)}(w + h) - f^{(n-1)}(w)}{h} = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} f(z) \frac{1}{h} \left(\frac{1}{(z - w - h)^n} - \frac{1}{(z - w)^n} \right) dz.$$

Using the formula for $\alpha, \beta \in \mathbb{C}$,

$$\alpha^n - \beta^n = (\alpha - \beta) \sum_{k=0}^{n-1} \alpha^{n-1-k} \beta^k$$

with $h \in D_{\delta}(0)$ and $z \in C_r(z_0)$ we get

$$\frac{1}{h} \left(\frac{1}{(z - w - h)^n} - \frac{1}{(z - w)^n} \right) = \sum_{k=0}^{n-1} \frac{1}{(z - w - h)^{n-k} (z - w)^{k+1}}.$$

From arithmetics we also get that

$$\lim_{h \rightarrow 0} \sup_{z \in C_r(z_0)} \left| \sum_{k=0}^{n-1} \frac{1}{(z - w - h)^{n-k} (z - w)^{k+1}} - \frac{n}{(z - w)^{n+1}} \right| = 0.$$

Note that $n/(z - w)^{n+1}$ is not part of the sum. From the above and Theorem 2.5 we get

$$\lim_{h \rightarrow 0} \frac{f^{(n-1)}(w + h) - f^{(n-1)}(w)}{h} = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} f(z) \frac{n}{(z - w)^{n+1}} dz,$$

which completes the proof of the theorem. □

Corollary 2.7 (Cauchy inequalities). Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \Omega$ and $r > 0$ be with $\overline{C}_r(z_0) \subset \Omega$. Then

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_{C_r(z_0)}}{r^n}$$

for all $n \geq 0$, where $\|f\|_{C_r(z_0)} = \sup \{|f(z)| : z \in C_r(z_0)\}$.

Proof. By Theorem 2.6 and Theorem 1.18 we get that

$$\begin{aligned}
 |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \\
 &\leq \frac{n!}{2\pi} \cdot \text{length}(C_r(z_0)) \cdot \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| \\
 &= n! \cdot r \cdot \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| \\
 &= \frac{n! \|f\|_{C_r(z_0)}}{r^n}.
 \end{aligned}$$

□

Remark 2.5. The following corollary might be one of the most surprising results in this course.

Corollary 2.8 (Liouville's theorem). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be entire and bounded. Then f is constant.*

Proof. According to Theorem 1.24 it suffices to show that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$ and since f is bounded let $C > 1$ be with $|f(z)| \leq C$ for all $z \in \mathbb{C}$. According to Cauchy's inequalities it follows that for every $R > 1$

$$|f'(z_0)| \leq \frac{\|f\|_{C_R(z_0)}}{R} \leq \frac{C}{R}.$$

Since this holds for each $R > 1$ it follows that $f'(z_0) = 0$ which completes the proof. □

Remark 2.6. We chose $R > 1$ because this way the results follow more naturally, but according to Cauchy's inequalities this of course holds for any $R > 0$ as well.

Corollary 2.9 (Fundamental theorem of algebra). *Let $p(z) = \sum_{k=0}^n a_k z^k$ be a nonconstant polynomial with complex coefficients. Then p has a root in $\mathbb{C}$.*

Proof. Assume by contradiction that p has no root in $\mathbb{C}$. Then $f := 1/p$ is an entire function. Since p is nonconstant we have $n \geq 1$ (so that $a_n \neq 0$) and thus

$$\liminf_{|z| \rightarrow \infty} |p(z)| \geq \liminf_{|z| \rightarrow \infty} |z|^n \left(|a_n| - \sum_{k=0}^{n-1} |a_k| \cdot |z|^{k-n} \right) = \infty.$$

Note that the power is $k - n$ because we factorized z^n . This means $|z|^{k-n} \xrightarrow{|z| \rightarrow \infty} 0$ which is why the equality at the end holds. In particular, this equality means that there exists $R > 1$ so that $|p(z)| \geq 1$ for all $z \in \mathbb{C} \setminus D_R(0)$ which implies that $|f(z)| \leq 1$ for all $z \in \mathbb{C} \setminus D_R(0)$. Since f is continuous and $\overline{D_R(0)}$ is compact, we have that $\|f\|_{\overline{D_R(0)}} < \infty$. We have shown that f is an entire and bounded function, so by Liouville's theorem it is constant, which implies that p is constant which is a contradiction to our assumption. This means that our assumption is false and p has a root in $\mathbb{C}$ which completes the proof. □

Corollary 2.10. *Let $n \geq 1$ and let $p(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree n with complex coefficients. Then there exists $w_1, \dots, w_n \in \mathbb{C}$ such that*

$$p(z) = a_n \prod_{k=1}^n (z - w_k).$$

Proof. We prove this corollary by induction. For the base case $n = 1$ we let $w_1 := a_0/a_1$ and then the claim holds. Suppose that the claim holds for $n - 1$ and let $p(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree n . From the fundamental theorem of algebra there exists $w_n \in \mathbb{C}$ so that $p(w_n) = 0$, and from polynomial long division there exists a polynomial $q(z) = \sum_{k=0}^{n-1} b_k z^k$ of degree $n - 1$ so that $p(z) = (z - w_n)q(z) + c$. We also get that $c = 0$ since

$$0 = p(w_n) = (w_n - w_n)q(w_n) + c = c$$

so $p(z) = (z - w_n)q(z)$. This implies that $a_n = b_{n-1}$. From this, and from the induction hypothesis, there exist $w_1, \dots, w_{n-1} \in \mathbb{C}$ so that

$$q(z) = a_n \prod_{k=1}^{n-1} (z - w_k).$$

This implies that

$$p(z) = (z - w_n)q(z) = (z - w_n)a_n \prod_{k=1}^{n-1} (z - w_k),$$

which completes the proof of this corollary. $\square$

2.4 The analyticity of holomorphic functions

Remark 2.7. We have already seen that an analytic function on an open set $\Omega \subset \mathbb{C}$ is holomorphic on that set. We now use Cauchy's formula to prove the converse.

Theorem 2.11. *Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, let $z_0 \in \Omega$, and let $R > 0$ be with $D_R(z_0) \subset \Omega$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z \in D_R(z_0)$, where $a_n := f^{(n)}(z_0)/n!$ for each $n \geq 0$. In particular, f is analytic on Ω .*

Proof. Fix some $z \in D_R(z_0)$ and let $|z - z_0| < r < R$. By Theorem 2.4

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z} dw.$$

For $w \in C_r(z_0)$ we have

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

and

$$\left| \frac{z - z_0}{w - z_0} \right| = \left| \frac{z - z_0}{r} \right| < 1.$$

Therefore by using the geometric series formula we get

$$\frac{1}{w - z} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

and the series converges uniformly for $w \in C_r(z_0)$. According to Theorem 2.5 we now obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n dw \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left((z - z_0)^n \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw \right). \end{aligned}$$

This shows that f is analytic. Moreover, by Theorem 2.6

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

for $n \geq 0$ which shows that the coefficients of the power series are what we proposed them to be which completes the proof. $\square$

Remark 2.8. Note that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is entire then it follows that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$, where $a_n := f^{(n)}(0)/n!$ for each $n \geq 0$.

Theorem 2.12. Let $\Omega \subset \mathbb{C}$ be a region and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Write $Z := \{z \in \Omega \mid f(z) = 0\}$, and suppose that Z has a limit point in Ω . Then $f(z) = 0$ for all $z \in \Omega$.

Proof. Denote by Ω_1 the set of limit points of Z in Ω . We will show that Ω_1 is an open subset of $\mathbb{C}$. Fix $z_0 \in \Omega_1$, and let $\epsilon > 0$ by with $D_\epsilon(z_0) \subset \Omega$. By Theorem 2.11 there exist $a_0, a_1, \dots \in \mathbb{C}$ so that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z \in D_\epsilon(z_0)$. Assume by way of contradiction that $a_n \neq 0$ for some $n \geq 1$. Then let $m = \min \{n \in \mathbb{N} \mid a_n \neq 0\}$ and denote $g(z) = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n$. Since $g(z_0) = a_m \neq 0$ and g is continuous there exists a $0 < \delta < \epsilon$ so that $g(z) \neq 0$ for all $z \in D_\delta(z_0)$. Since z_0 is a limit point of z there exists a point $w \in D_\delta(z_0)$ so that $f(w) = 0$. We now see that since $w \in D_\delta(z_0)$ and $w \neq z_0$ that $(w - z_0)^m g(w) \neq 0$. However, we also have that

$$0 = f(w) = \sum_{n=m}^{\infty} a_n (w - z_0)^n = (w - z_0)^m g(w).$$

This contradiction implies that $a_n = 0$ for all $n \geq 0$, which means that $f(z) = 0$ for all $z \in D_\epsilon(z_0)$. This implies that $D_\epsilon(z_0) \subset Z$, which very elegantly implies that $D_\epsilon(z_0) \subset \Omega_1$, which means that Ω_1 is indeed an open subset of $\mathbb{C}$.

We now assume by contradiction that $\Omega_2 := \Omega \setminus \Omega_1$ is nonempty, and fix $z_0 \in \Omega_2$. Since z_0 is not a limit point of Z there exists $\epsilon > 0$ so that $D_\epsilon(z_0) \subset \Omega$ and $f(z) \neq 0$ for all $z \in D_\epsilon(z_0) \setminus \{z_0\}$. This also elegantly implies that the point in $D_\epsilon(z_0)$ are not limit points of Z , and thus $D_\epsilon(z_0) \subset \Omega_2$ which means that Ω_2 is also an open subset of $\mathbb{C}$.

Since Ω is a region, and since Ω is the disjoint union of two open sets Ω_1 and Ω_2 , this implies that either $\Omega_1 = \emptyset$ or $\Omega_2 = \emptyset$. Since by assumption $\Omega_1 \neq \emptyset$ we get that $\Omega_2 = \emptyset$ and so $\Omega_1 = \Omega$. This means that every $z \in \Omega$ is a limit point of Z , and since f is continuous it follows that $\Omega = Z$ which completes the proof. $\square$

Corollary 2.13. Let $\Omega \subset \mathbb{C}$ be a region and let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic. Set $A := \{z \in \Omega \mid f(z) = g(z)\}$, and suppose that A has a limit point in Ω . Then $f(z) = g(z)$ for all $z \in \Omega$.

Proof. The proof follows directly from applying Theorem 2.12 on the function $f - g$. $\square$

Definition 2.3 (Analytic continuation). Let $\Omega_1, \Omega_2 \in \mathbb{C}$ be regions with $\Omega_1 \subset \Omega_2$. For $i = 1, 2$ let $f_i: \Omega_i \rightarrow \mathbb{C}$ be holomorphic, and suppose that $f_1 = f_2$ on Ω_1 . We say that f_2 is an analytic continuation of f_1 into the region Ω_2 .

Remark 2.9. From Theorem 2.13 it follows that the analytic continuation of f_1 into Ω_2 is unique.

2.5 Further applications

Theorem 2.14 (Morera's theorem). Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be continuous, and suppose that $\int_{\partial\Delta} f dz = 0$ for every closed triangle $\Delta \subset \Omega$. Then f is holomorphic on Ω .

Proof. Since we need to show that f is holomorphic at every $z \in \Omega$, and any $z \in \Omega$ is contained in a sufficiently small epsilon ball, it suffices to show that f is holomorphic on every open convex subset of Ω . Let $V \subset \Omega$ be an open convex set. Fix $z_0 \in V$, and for $z \in V$ set

$$F(z) := \int_{[z_0, z]} f(w) dw.$$

In the exact same process from Theorem 2.2 we can show that F is a primitive for f on V , and in particular it is holomorphic on V . By Theorem 2.5 it follows that complex derivatives of holomorphic functions are holomorphic. This means that $f = F'$ is holomorphic on V which completes the proof. $\square$

Definition 2.4 (Uniform convergence on compact subsets). Let $\Omega \subset \mathbb{C}$ and $f, f_1, f_2, \dots : \Omega \rightarrow \mathbb{C}$ be given. We say that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on compact subsets of Ω if for every compact $K \subset \Omega$ we have $\|f - f_n\|_K \xrightarrow{n \rightarrow \infty} 0$.

Remark 2.10. Suppose we also know that Ω is open and that $f_1, f_2, \dots$ are continuous on Ω . Then f is also continuous on Ω .

Proposition 2.15. Let $\Omega \subset \mathbb{C}$ and $f, f_1, f_2, \dots : \Omega \rightarrow \mathbb{C}$ be given. Suppose that f_n is holomorphic on Ω for each $n \geq 1$, and that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on compact subsets of Ω . Then f is also holomorphic on Ω .

Proof. According to Remark 2.10 it follows that f is continuous. Let $\Delta \subset \Omega$ be a closed triangle. Since Δ is compact, we have $\|f - f_n\|_{\Delta} \xrightarrow{n \rightarrow \infty} 0$. From Theorem 2.5 we obtain that

$$\int_{\partial\Delta} f_n dz \xrightarrow{n \rightarrow \infty} \int_{\partial\Delta} f dz.$$

However, since f_n is holomorphic for $n \geq 1$, it follows that every term in the sequence is 0 which implies that $\int_{\partial\Delta} f dz = 0$. Thus from Morera's theorem it follows that f is holomorphic on Ω which completes the proof. $\square$

Remark 2.11. This proposition is clearly false in the case of real analysis. This is because by the Weierstrass approximation theorem, any continuous function on a compact interval can be uniformly approximated by polynomials (which are holomorphic), but not every continuous function on a compact interval is holomorphic when extended to the complex plane.

Example 2.3 (Riemann zeta function). For $r > 0$ and $z \in \mathbb{C}$ we define $z := e^{z \log r}$, where $\log$ is the standard real logarithm function. Since $z \mapsto e^z$ is entire, it follows from arithmetic operations that $z \mapsto r^z$ is also entire. For $t > 0$ we define the set

$$H_t := \{z \in \mathbb{C} : \operatorname{Re}(z) > t\}.$$

Given $t > 1$ and $z \in H_t$ we have that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right| = \sum_{n=1}^{\infty} \left| e^{-z \log n} \right| = \sum_{n=1}^{\infty} e^{-\operatorname{Re}(z) \log n} \leq \sum_{n=1}^{\infty} e^{-t \log n} = \sum_{n=1}^{\infty} \frac{1}{n^t} < \infty.$$

From this follows that the function $\zeta : H_1 \rightarrow \mathbb{C}$ with

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is well defined. The function ζ is called the Riemann zeta function. We can also show that it is holomorphic on H_1 . Let $t > 1$ be given. As we saw previously for $z \in H_t$ and $N \geq 1$ we have

$$\left| \zeta(z) - \sum_{n=1}^N \frac{1}{n^z} \right| = \left| \sum_{n=N+1}^{\infty} \frac{1}{n^z} \right| \leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n^z} \right| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^t} \xrightarrow{N \rightarrow \infty} 0$$

where the last step follows from $t > 1$. This shows the function sequence $\left\{ z \mapsto \sum_{n=1}^N \frac{1}{n^z} \right\}_{N \geq 1}$ converges uniformly to ζ on H_t . Thus, since $z \mapsto \sum_{n=1}^N \frac{1}{n^z}$ is entire for all $N \geq 1$ and from Theorem 2.15 we get that ζ is holomorphic on H_t . Since this holds for all $t > 1$, we get that ζ is holomorphic on H_1 .

Remark 2.12. There are similar theorems about convergence of sequences of derivatives.

Theorem 2.16. Let $\Omega \subset \mathbb{C}$ be open and let $f, f_1, f_2, \dots : \Omega \rightarrow \mathbb{C}$ be holomorphic. Suppose that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on compact subsets of Ω . Then for each $k \geq 1$ we have $f_n^{(k)} \xrightarrow{n \rightarrow \infty} f^{(k)}$ uniformly on compact subsets of Ω .

Proof. To be added. □

Proposition 2.17. Let $\Omega \subset \mathbb{C}$ be open and let $F : \Omega \times [0, 1] \rightarrow \mathbb{C}$ be continuous on $\Omega \times [0, 1]$. Suppose that for each $s \in [0, 1]$ the function $z \mapsto F(z, s)$ is holomorphic on Ω . Let $f : \Omega \rightarrow \mathbb{C}$ be with

$$f(z) := \int_0^1 F(z, s) ds$$

for $z \in \Omega$, then f is holomorphic on Ω .

Proof. Using Theorem 2.15. To be added. □

Example 2.4 (Gamma function). For each $0 < \epsilon < 1$ let $f_\epsilon : \mathbb{C} \rightarrow \mathbb{C}$ be with

$$f_\epsilon(z) := \int_\epsilon^{1/\epsilon} e^{-t} t^{z-1} dt,$$

where $t^{z-1} := e^{(z-1) \log t}$ for $t > 0$. By Theorem 2.17 it follows that f_ϵ is entire.

Set $H_0 := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$, and let $\Gamma : H_0 \rightarrow \mathbb{C}$ be such that

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt.$$

The function Γ which is called the gamma function, is well defined. Since it can be shown that $f_{1/n} \xrightarrow{n \rightarrow \infty} \Gamma$ uniformly on compact subsets of H_0 , it follows by Theorem 2.15 that Γ is holomorphic on H_0 .

2.6 Harmonic functions

Definition 2.5 (Harmonic function). Let $U \subset \mathbb{C}$ be a region and let $u : U \rightarrow \mathbb{R}$ be a real-valued function with 2nd order partial derivatives. Then u is said to be harmonic on U if it satisfies the Laplace equation:

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

at any point $p \in U$.

Remark 2.13. We say that u is harmonic at $p \in U$ if it satisfies the Laplace equation at u . In some sources, we say that u is harmonic on U if it is harmonic at all $u \in \overline{U}$.

Remark 2.14. The Laplace equation is one of the most important partial differential equations of mathematical physics.

Proposition 2.18. *Let $U \subset \mathbb{C}$ be a region and let $f: U \rightarrow \mathbb{C}$ so that $f = u + iv$ be holomorphic. Then u and v are harmonic functions.*

Proof. Since f is holomorphic, by Theorem 2.11 and Theorem 1.16 it is infinitely complex differentiable, and thus u and v have continuous 2nd derivatives. From the Cauchy–Riemann equations we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows that

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) \\ &= 0. \end{aligned}$$

The same argument holds for v , which completes the proof. $\square$

Definition 2.6 (Harmonic conjugate). Let $U \subset \mathbb{C}$ be a region and let $u: U \rightarrow \mathbb{R}$ be harmonic. A harmonic function $v: U \rightarrow \mathbb{R}$ is called a harmonic conjugate of u if $f = u + iv$ is holomorphic.

Remark 2.15. Note that if v is a harmonic conjugate of u and a is a complex number then $v + a$ is also a harmonic conjugate of u since

$$u + i(v + a) = f + ia$$

is holomorphic. On the other hand, if v and w are both harmonic conjugates of u , then we have

$$i(w - v) = (u + iv) - (u + iw) = f - g$$

which is holomorphic as the difference of two holomorphic functions. Since $i(w - v)$ is purely imaginary and holomorphic, it must be constant. This implies that $w = v + a$ for some complex number $a \in \mathbb{C}$. In conclusion, harmonic functions are unique up to addition of a constant.

Example 2.5. It is clear that $u = xy$ is harmonic since its second partial derivatives are continuous, and we have

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In order to find the harmonic conjugate we first need to solve the Cauchy–Riemann equations

$$\begin{aligned} v_y &= x, \\ v_x &= -y. \end{aligned}$$

In order to solve these equations we see that

$$v(x, y) = \frac{y^2}{2} + h(x) + c$$

where $h(x)$ is some function in x , and c is some constant. Plugging $v(x, y)$ into the second equation gives $h'(x) = -x$ which implies

$$v(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + C.$$

For simplicity we will only consider $v(x, y)$ with $c = 0$. We notice that if $f = u + iv$, then we actually have

$$f(z) = i\frac{z^2}{2} + c.$$

Proposition 2.19. *Let U be an open rectangle with sides parallel to the axes. Then every harmonic function u on U has a harmonic conjugate.*

Proof. Let $(x_0, y_0) \in U$. We start by integrating the first Cauchy–Riemann equation with respect to y :

$$v(x, y) = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt + h(x),$$

where $h(x)$ is the constant of integration (which may depend on x). We then proceed to plug this into the second Cauchy–Riemann equation:

$$\begin{aligned} \frac{\partial u}{\partial y}(x, y) &= -\frac{\partial}{\partial x} \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt - h'(x) \\ &= -\int_{y_0}^y \frac{\partial^2 u}{\partial x^2}(x, t) dt - h'(x) \\ &= \int_{y_0}^y \frac{\partial^2 u}{\partial y^2}(x, t) dt - h'(x) \\ &= \frac{\partial u}{\partial y}(x, y) - \frac{\partial u}{\partial y}(x, y_0) - h'(x). \end{aligned}$$

Hence

$$h'(x) = -\frac{\partial u}{\partial y}(x, y_0).$$

Integrating both sides with respect to x gives

$$h(x) = -\int_{x_0}^x \frac{\partial u}{\partial y}(s, y_0) ds,$$

up to addition of a constant. Thus,

$$v(x, y) = \int_{y_0}^y \frac{\partial u}{\partial x}(x, t) dt - \int_{x_0}^x \frac{\partial u}{\partial y}(s, y_0) ds,$$

up to a constant. □

Proposition 2.20. *Let U be an open disk. Then every harmonic function u on U has a harmonic conjugate.*

Proof. We can use the same proof from Theorem 2.19, except we must choose (x_0, y_0) to be the center of the disc. That is because if we choose another point, the rectangle we integrate on could intersect with the space outside the disc. □

3 Meromorphic functions and the logarithm

3.1 Zeros and singularities

Definition 3.1 (Zero). Let $\Omega \subset \mathbb{C}$ be open and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. A number $z_0 \in \Omega$ is said to be a zero of f if $f(z_0) = 0$.

Proposition 3.1. Let $\Omega \subset \mathbb{C}$ be a region, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, suppose that f does not vanish identically in Ω , and let z_0 be a zero of f . Then there exists a unique integer $m \geq 1$ for which there exists a holomorphic $g: \Omega \rightarrow \mathbb{C}$ so that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for each $z \in \Omega$.

Proof. Since a holomorphic function is analytic and Ω is a region, there exists $\epsilon > 0$ and $\{a_i\}_{i=0}^\infty \subset \mathbb{C}$ so that $D_\epsilon(z_0) \subset \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in D_\epsilon(z_0)$. Since f does not vanish identically in Ω , by Theorem 2.12 it does not vanish identically in $D_\epsilon(z_0)$. Since f is analytic and $f(z_0) = 0$ it follows elegantly that there exists $m \geq 1$ so that $a_m \neq 0$ and $a_n = 0$ for $0 \leq n < m$. We now define $g: \Omega \rightarrow \mathbb{C}$ by

$$g(z) := \begin{cases} (z - z_0)^{-m} f(z), & z \in \Omega \setminus \{z_0\} \\ a_m, & z = z_0 \end{cases}.$$

It is clear that f is holomorphic on $\Omega \setminus \{z_0\}$. By definition we have that

$$g(z) = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n$$

for $z \in D_\epsilon(z_0)$. This means it is analytic and thus holomorphic at z_0 so it is holomorphic on Ω and $f(z) = (z - z_0)^m g(z)$ for $z \in \Omega$. We also note that $g(z_0) = a_m \neq 0$ so it satisfies all required properties.

We still need to show that m is unique. Let $k \geq 1$ be an integer for which there exists a holomorphic $h: \Omega \rightarrow \mathbb{C}$ so that $h(z_0) \neq 0$ and $f(z) = (z - z_0)^k h(z)$ for each $z \in \Omega$.

Assume by contradiction that $m < k$. Then we obtain $g(z) = (z - z_0)^{k-m} h(z)$ for $z \in \Omega \setminus \{z_0\}$. As $z \rightarrow z_0$ we get from continuity of g that $g(z_0) = 0$ which is a contradiction to $g(z_0) \neq 0$. A similar argument shows that $k < m$ is not possible, which completes the proof. $\square$

Definition 3.2 (Isolated singularity). Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. In this case we say that f has an isolated singularity at z_0 .

Remark 3.1 (Removable singularity). If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω then we say that z_0 is a removable singularity of f .

Definition 3.3 (Punctured disc). Given $z_0 \in \mathbb{C}$ and $r > 0$ we denote $D'_r(z_0)$ the punctured open disc with center z_0 and radius r . That is

$$D'_r(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.$$

Theorem 3.2 (Riemann's removable singularity theorem). Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. Suppose that f is bounded in $D'_r(z_0)$ for some $r > 0$. Then z_0 is a removable singularity of f .

Proof. Let $g: \Omega \rightarrow \mathbb{C}$ be such that

$$g(z) := \begin{cases} 0, & z = z_0 \\ (z - z_0)^2 f(z), & z \in \Omega \setminus \{z_0\} \end{cases}.$$

Since f is bounded in $D'_r(z_0)$ we get that

$$\lim_{h \rightarrow 0} \frac{g(z_0 + h) - g(z_0)}{h} = \lim_{h \rightarrow 0} h f(z_0 + h) = 0,$$

which means that g is holomorphic at z_0 with $g'(z_0) = 0$. Since g is also holomorphic on $\Omega \setminus \{z_0\}$, we get in total that g is holomorphic on Ω . By Theorem 2.11 we get that g is analytic on Ω and since $g(z_0) = g'(z_0) = 0$, there exists $\epsilon > 0$ with $D_\epsilon(z_0) \subset \Omega$ and $a_2, a_3, \dots \in \mathbb{C}$ so that $g(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^n$ for $z \in D_\epsilon(z_0)$. thus for $z \in D'_\epsilon(z_0)$ we have

$$f(z) = (z - z_0)^{-2} g(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2} = \sum_{n=0}^{\infty} a_{n+2} (z - z_0)^n.$$

The function on the right hand side is a holomorphic extension of f to $D_\epsilon(z_0)$ which means that by setting $f(z_0) := a_2$ we get the desired result which completes the proof. $\square$

Remark 3.2. The above theorem is sometimes called Riemann's theorem, and it actually states that the following are equivalent

- (1) f is holomorphically extendable over z_0 .
- (2) f is continuously extendable over z_0 .
- (3) There exists a neighbourhood of a in which f is bounded.
- (4) $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$.

Definition 3.4 (Pole). Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. We say that f has a pole at z_0 if there exists $\epsilon > 0$ such that $D_\epsilon(z_0) \subset \Omega$ and $f(z) \neq 0$ for $z \in D'_\epsilon(z_0)$, and the function

$$g(z) := \begin{cases} \frac{1}{f(z)}, & \text{for } z \in D'_\epsilon(z_0) \\ 0, & \text{for } z = z_0 \end{cases}$$

is holomorphic on $D_\epsilon(z_0)$.

Remark 3.3. Note that if f has a pole at z_0 then $\lim_{z \rightarrow z_0} |f(z)| = \infty$. We will see later that this is an if and only if statement.

Proposition 3.3. Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic, and suppose that f has a pole at z_0 . Then there exists a unique integer $m \geq 1$ for which there exists $\delta > 0$ with $D_\delta(z_0) \subset \Omega$ and a holomorphic $g: D_\delta(z_0) \rightarrow \mathbb{C}$ such that $g(z) \neq 0$ and $f(z) = (z - z_0)^{-m} g(z)$ for $z \in D'_\delta(z_0)$.

Remark 3.4. The integer m is called the order of the pole which f has at z_0 .

Remark 3.5 (Simple pole). If the order of pole is $m = 1$, we say that it is a simple pole.

Proof. To be added. $\square$

Proposition 3.4. Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic, and suppose that f has a pole of order m at z_0 . Then there exists $\epsilon > 0$ with $D_\epsilon(z_0) \subset \Omega$ and $a_{-m}, a_{-m+1}, \dots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$, and a holomorphic $h: D_\epsilon(z_0) \rightarrow \mathbb{C}$, such that

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + h(z)$$

for $z \in D'_\epsilon(z_0)$.

Remark 3.6. The sum $\frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)}$ is called the *principle part* of f at the pole z_0 , and the coefficient a_{-1} is called the residue of f at that pole. We shall write $\text{Res}(f, z_0)$ instead of a_{-1} . The importance of the residue comes from the fact that all the other terms in the principle part have primitives in a deleted neighbourhood of z_0 . This will be extremely helpful for us later when we will talk about the residue theorem.

Proof. Since f has a pole of order m at z_0 , there exists $\epsilon > 0$ with $D_\epsilon(z_0) \subset \Omega$ and a holomorphic $g: D_\epsilon(z_0) \rightarrow \mathbb{C}$ such that $g(z) \neq 0$ and $f(z) = (z-z_0)^{-m}g(z)$ for $z \in D'_\epsilon(z_0)$. Since g is holomorphic by Theorem 2.11 there exist $b_0, b_1, \dots \in \mathbb{C}$ so that

$$g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n.$$

Since $g(z_0) \neq 0$ we get that $b_0 \neq 0$. For $-m \leq n \leq -1$ set $a_n := b_{m+n}$, and let $h: D_\epsilon(z_0) \rightarrow \mathbb{C}$ be with

$$h(z) = \sum_{n=0}^{\infty} b_{m+n}(z-z_0)^n.$$

Since h is a power series it is holomorphic. We also have for each $z \in D'_\epsilon(z_0)$ we have

$$\begin{aligned} f(z) &= (z-z_0)^{-m}g(z) \\ &= \sum_{n=0}^{\infty} b_n(z-z_0)^{n-m} \\ &= \sum_{n=-m}^{-1} b_{m+n}(z-z_0)^n + \sum_{n=0}^{\infty} b_{m+n}(z-z_0)^n \\ &= \sum_{n=-m}^{-1} a_n(z-z_0)^n + h(z). \end{aligned}$$

Since $a_{-m} = b_0 \neq 0$, this completes the proof of this proposition. $\square$

Lemma 3.5 (Residue calculation lemma). Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic, suppose that f has a pole of order m at z_0 , and set $g(z) := (z-z_0)^m f(z)$ for $z \in \Omega \setminus \{z_0\}$. Then,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} g^{(m-1)}(z).$$

In particular, if z_0 is a simple pole then,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z-z_0)f(z).$$

Proof. Since f has a pole of order m at z_0 , by Theorem 3.4 there exists $\epsilon > 0$ with $D_\epsilon(z_0) \subset \Omega$ and $\{a_n\}_{n=-m}^{\infty} \subset \mathbb{C}$ so that

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n$$

for $z \in D'_\epsilon(z_0)$. We then have by definition that

$$g(z) = \sum_{n=0}^{\infty} a_{n-m}(z - z_0)^n.$$

Hence,

$$\lim_{z \rightarrow z_0} g^{m-1}(z) = \lim_{z \rightarrow z_0} \sum_{n=m-1}^{\infty} a_{n-m} \frac{n!}{(n-m+1)!} (z - z_0)^{n-m+1} = a_{-1}(m-1)!,$$

which completes the proof. $\square$

Proposition 3.6. *Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. Then f has a pole at z_0 if and only if $|f(z)| \xrightarrow{z \rightarrow z_0} \infty$.*

Proof. Suppose that f has a pole at z_0 . By Theorem 3.3 there exists $m \geq 1$, $\epsilon > 0$ and a holomorphic function $g: D_\epsilon(z_0) \rightarrow \mathbb{C}$, so that $D_\epsilon(z_0) \subset \Omega$, $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-m}g(z)$ for $z \in D'_\epsilon(z_0)$, which immediately implies that $|f(z)| \xrightarrow{z \rightarrow z_0} \infty$.

Suppose that $|f(z)| \xrightarrow{z \rightarrow z_0} \infty$. Then there exists $\epsilon > 0$ so that $|f(z)| \geq 1$ for $z \in D'_\epsilon(z_0)$, which implies that the function $1/f$ is holomorphic and bounded on $D'_\epsilon(z_0)$. By Theorem 3.2 this means that there exists $\alpha \in \mathbb{C}$ so that the function

$$g(z) := \begin{cases} \frac{1}{f(z)}, & \text{for } z \in D'_\epsilon(z_0) \\ \alpha, & \text{for } z = z_0 \end{cases}$$

is holomorphic on $D_\epsilon(z_0)$. Since $|f(z)| \xrightarrow{z \rightarrow z_0} \infty$ we must have $\alpha = 0$. This shows by definition that f has a pole at z_0 which completes the proof. $\square$

Definition 3.5 (Essential singularity). Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. We say that z_0 is an essential singularity if it is not a removable singularity and not a pole.

Theorem 3.7 (Casorati–Weierstrass theorem). *Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic. Suppose that z_0 is an essential singularity of f , and let $\epsilon > 0$ be with $D_\epsilon(z_0) \subset \Omega$. Then $f(D'_\epsilon(z_0))$ is dense in $\mathbb{C}$.*

Proof. Assume by contradiction that $f(D'_\epsilon(z_0))$ is not dense in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ and $\delta > 0$ so that $|f(z) - w| \geq \delta$ for all $z \in D'_\epsilon(z_0)$. This allows us to define $g: D'_\epsilon(z_0) \rightarrow \mathbb{C}$ by

$$g(z) := \frac{1}{f(z) - w}.$$

Since g is holomorphic and bounded on $D'_\epsilon(z_0)$, by Theorem 3.2 it follows that g can be extended to a holomorphic function on $D_\epsilon(z_0)$. We continue to denote this extension by g .

If $g(z_0) \neq 0$ then there exists $0 < \eta < \epsilon$ so that $|g(z)| \geq \eta$ for all $z \in D_\eta(z_0)$, which means that for all $z \in D'_\eta(z_0)$ we have

$$|f(z)| \leq |f(z) - w| + |w| \leq \eta^{-1} + |w|.$$

Then again by Theorem 3.2 it follows that z_0 is a removable singularity of f , in contradiction to the assumption. This implies that $g(z_0) = 0$ which means that $g(z) \xrightarrow{z \rightarrow z_0} 0$. By definition of g this implies that $|f(z) - w| \xrightarrow{z \rightarrow z_0} \infty$, which clearly implies that $|f(z)| \xrightarrow{z \rightarrow z_0} \infty$. Hence by Theorem 3.6 we have that z_0 is a pole of f , which still forms a contradiction since we have shown that z_0 is a removable singularity of g which completes the proof. $\square$

3.2 Meromorphic functions

Definition 3.6 (Meromorphic function). Let $\Omega \subset \mathbb{C}$ be open, let $A \subset \Omega$, and let $f: \Omega \setminus A \rightarrow \mathbb{C}$ be such that,

- (1) A has no limit points in Ω ;
- (2) f is holomorphic on $\Omega \setminus A$;
- (3) f has a pole at each point of A .

Then we say that f is meromorphic on Ω .

Remark 3.7. We can always choose $A = \emptyset$, hence every holomorphic function is meromorphic. Note that if f is a nonzero meromorphic function on Ω , then $1/f$ is also meromorphic on Ω (albeit with a different subset of problematic points). A zero of f is a pole of $1/f$, and a pole of f is a zero of $1/f$.

Remark 3.8. Since A has no limit point in Ω , every compact subset of Ω contains at most finitely many points of A . Since we can cover Ω with a countable amount of compact balls around the rational points, it follows that $|A| \leq \aleph_0$.

Definition 3.7 (Functions at ∞). Let $\Omega \subset \mathbb{C}$ be open, and suppose that $\mathbb{C} \setminus \overline{D}_M(0) \subset \Omega$ for some $M > 1$. Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, and for $z \in D'_{1/M}(0)$ set $F(z) := f(1/z)$. We see that f is well defined and holomorphic on $D'_{1/M}(0)$, and that 0 is an isolated singularity of F . We say that f has a removable singularity at ∞ , or that it is holomorphic at ∞ if F has a removable singularity at 0. Similarly, we say that f has a pole at ∞ if F has a pole at 0. Finally, we say that f has an essential singularity at ∞ if F has an essential singularity at 0.

Definition 3.8 (Meromorphic on $\mathbb{C} \cup \{\infty\}$). Let f be meromorphic on $\mathbb{C}$, and suppose that f has at most finitely many poles. If in addition f is holomorphic at ∞ or has a pole at ∞ , we say that f is meromorphic in the *extended complex plane*.

Example 3.1. Let $p, q: \mathbb{C} \rightarrow \mathbb{C}$ be polynomials with complex coefficients, suppose that q is not the zero polynomial, and let $\{z_i\}_{i=1}^n$ be the zeros of q . Let $r: \mathbb{C} \setminus \{z_i\}_{i=1}^n \rightarrow \mathbb{C}$ be such that $r(z) := p(z)/q(z)$. Functions of this form are called rational functions. We will show that r is meromorphic in the extended complex plane.

We see that r has a finite amount of poles, and that if $\deg(q) \geq \deg(p)$, there exists $\alpha \in \mathbb{C}$ such that $\lim_{|z| \rightarrow \infty} |r(z)| = \alpha$. Thus, in this case from Theorem 3.2 it follows that r is holomorphic at ∞ .

If $\deg(q) < \deg(p)$ then we have that $\lim_{|z| \rightarrow \infty} |r(z)| = \infty$ so from Theorem 3.6 it follows that r has a pole at ∞ .

This implies that r is meromorphic in the extended complex plane.

Proposition 3.8. *Let f be meromorphic in the extended complex plane. Then f is a rational function.*

Remark 3.9. Coupled with the previous example, this shows that f is meromorphic in the extended complex plane if and only if it is a rational function.

Proof. Let $z_1, z_2, \dots, z_n$ be the distinct poles of f in $\mathbb{C}$. By Theorem 3.4 for each $1 \leq k \leq n$ there exist $\epsilon_k > 0$, a polynomial $p_k(z) \in \mathbb{C}[z]$, and a holomorphic function $g_k: D_{\epsilon_k}(z_k) \rightarrow \mathbb{C}$ so that

$$f(z) = p_k \left(\frac{1}{z - z_k} \right) + g_k(z)$$

for all $z \in D'_{\epsilon_k}(z_k)$. Similarly for $f(1/z)$ there exists $\epsilon_0 > 0$, a polynomial $p_0(z) \in \mathbb{C}[z]$, and a holomorphic function $g_0: D_{\epsilon_0}(0) \rightarrow \mathbb{C}$ so that

$$f(1/z) = p_0(1/z) + g_0(z)$$

for all $z \in D'_{\epsilon_0}(0)$. Set $M := 1/\epsilon_0$ and we now get that

$$f(z) = p_0(z) + g_0(1/z)$$

for all $z \in \mathbb{C} \setminus \overline{D}_M(0)$. Set $\Omega := \mathbb{C} \setminus \{z_1, \dots, z_n\}$ and let $h: \Omega \rightarrow \mathbb{C}$ be with

$$h(z) := f(z) - p_0(z) - \sum_{k=1}^n p_k \left(\frac{1}{z - z_k} \right)$$

for $z \in \Omega$. We will now show that h can be extended to an entire function. Let $0 < \delta < \epsilon_1$ be such that $z_k \notin D_\delta(z_1)$ for all $2 \leq k \leq n$. Then by combining previous results we have that for $z \in D'_\delta(z_1)$,

$$h(z) = g_1(z) - p_0(z) - \sum_{k=2}^n p_k \left(\frac{1}{z - z_k} \right).$$

Since the function on the right hand side is holomorphic on $D_\delta(z_1)$ it follows that h has a removable singularity at z_1 . Similarly all of its singularities are removable and so we may regard it as an entire function.

We now show that h is entire. Let $M_0 \geq M$ be such that $M_0 > |z_k| + 1$ for all $z_1, \dots, z_k$. For $z \in \mathbb{C}$ with $|z| \geq M_0$

$$|z - z_k| \geq |z| - |z_k| > 1 \implies \left| p_k \left(\frac{1}{z - z_k} \right) \right| \leq \|p_k\|_{D_1(0)}.$$

This implies that for $z \in \mathbb{C}$ with $|z| \geq 2M_0$,

$$\begin{aligned} |h(z)| &\leq |f(z) - p_0(z)| + \sum_{k=1}^n \left| p_k \left(\frac{1}{z - z_k} \right) \right| \\ &= |g_0(1/z)| + \sum_{k=1}^n \left| p_k \left(\frac{1}{z - z_k} \right) \right| \leq \|g_0\|_{D_{1/(2M)}(0)} + \sum_{k=1}^n \|p_k\|_{D_1(0)}, \end{aligned}$$

which means that h is bounded on $\mathbb{C} \setminus D_{2M_0}(0)$. Since h is entire this implies that it is bounded on $\mathbb{C}$. From Liouville's theorem it now follows that h is constant. After recalling that

$$h(z) := f(z) - p_0(z) - \sum_{k=1}^n p_k \left(\frac{1}{z - z_k} \right)$$

we notice that this completed the proof. □

3.3 Homotopies and simply connected domains

Remark 3.10. In this subsection, paths are not automatically assumed to be piecewise-smooth. A path is simply a continuous map γ from $[a, b]$ to $\mathbb{C}$.

Definition 3.9 (Homotopy of paths). Let $\Omega \subset \mathbb{C}$ be open, let $[a, b]$ be an interval, and let $\gamma_0, \gamma_1: [a, b] \rightarrow \Omega$ be paths. A homotopy between γ_0 and γ_1 is a continuous map $H: [a, b] \times [0, 1] \rightarrow \Omega$ such that for all $s \in [a, b]$ and $t \in [0, 1]$

$$H(s, 0) = \gamma_0(s) \text{ and } H(s, 1) = \gamma_1(s) \text{ and } H(a, t) = \gamma_0(a) \text{ and } H(b, t) = \gamma_0(b).$$

If there exists a homotopy between γ_0 and γ_1 , we say that they are homotopic and denote $\gamma_0 \sim \gamma_1$.

Remark 3.11. Note that if $\gamma_0 \sim \gamma_1$ then $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$.

Theorem 3.9. Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, and let $\gamma_0, \gamma_1: [a, b] \rightarrow \Omega$ be piecewise-smooth paths. Suppose that $\gamma_0 \sim \gamma_1$ (path homotopic) in Ω , then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof. To be added. □

Definition 3.10 (Simply connected space). A region $\Omega \subset \mathbb{C}$ is said to be simply connected if any two paths $\gamma_0, \gamma_1: [a, b] \rightarrow \Omega$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$ are homotopic.

Definition 3.11 (Homeomorphic spaces). Two subsets $A, B \subset \mathbb{C}$ are said to be homeomorphic if there exists a bijection $h: A \rightarrow B$ such that both h and h^{-1} are continuous.

Remark 3.12. We see that if A and B are homeomorphic then A is simply connected if and only if B is simply connected. This means that simple connectivity is a topological property.

Example 3.2. Any convex nonempty open $\Omega \subset \mathbb{C}$ is simply connected. For $\gamma_0, \gamma_1: [a, b] \rightarrow \Omega$ with common endpoints we can define the homotopy $H: [a, b] \times [0, 1] \rightarrow \Omega$ by

$$H(s, t) := (1 - s)\gamma_0(t) + s\gamma_1(t).$$

Example 3.3. The set $\Omega := \mathbb{C} \setminus (-\infty, 0]$ is simply connected.

Example 3.4. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

Theorem 3.10 (Cauchy's theorem in a simply connected region). Let $\Omega \subset \mathbb{C}$ be a simply connected region, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then f has a primitive on Ω , and $\int_{\gamma} f(z)dz = 0$ for every piecewise-smooth closed path $\gamma: [a, b] \rightarrow \Omega$.

Proof. By Theorem 1.22 it suffices to show that f has a primitive F on Ω . Fix $z_0 \in \Omega$ and let $F: \Omega \rightarrow \mathbb{C}$ be with

$$F(z_1) := \int_{\gamma} f(z)dz$$

where $\gamma: [0, 1] \rightarrow \Omega$ is any piecewise-smooth path from z_0 to z_1 . Since Ω is region by Theorem 1.25 such a path must exist. Since Ω is simply connected, any piecewise-smooth path $\sigma: [0, 1] \rightarrow \Omega$ from z_0 to z_1 is path homotopic to γ in Ω . Thus, since f is holomorphic from Theorem 3.9 we get that

$$\int_{\gamma} f(z)dz = \int_{\sigma} f(z)dz,$$

which shows that F is well defined.

We now only need to show that F is a primitive of f . Let $z_1 \in \Omega$. Since f is holomorphic on Ω , it is in particular continuous at z_1 , so for any $\epsilon > 0$ there exists $\delta > 0$ with $D_{\delta}(z_1)$ so that for any $z \in D_{\delta}(z_1)$ we have that $|f(z) - f(z_1)| < \epsilon$. Let $0 \neq h \in D_{\delta}(0)$, let $\gamma_0: [0, 1] \rightarrow \Omega$ be the path from z_0 to $z_1 + h$ and let $\gamma_1: [0, 1] \rightarrow \Omega$ be the path that travels on γ_0 , then continues on $[z_1 + h, z_1]$. More rigorously we define

$$\gamma_1(t) := \begin{cases} \gamma_0(2t), & \text{for } t \in [0, 0.5] \\ z_1 + 2(1 - t)h, & \text{for } t \in [0.5, 1] \end{cases}.$$

By definition of F ,

$$F(z_1 + h) - F(z_1) = \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{[z_1, z_1 + h]} f(z)dz.$$

Thus, as in the proof for Cauchy's theorem in a convex set (Theorem 2.2), since $[z_1, z_1 + h] \subset D_\delta(z_1)$,

$$\left| \frac{F(z_1 + h) - F(z_1)}{h} - f(z_1) \right| = \left| \frac{1}{h} \int_{[z_1, z_1 + h]} f(z) - f(z_1) dz \right| < |h|^{-1} \text{length}([z_1, z_1 + h]) \epsilon = \epsilon.$$

This shows that F is holomorphic at z_1 with $F'(z_1) = f(z_1)$, which completes the proof. $\square$

Remark 3.13. Example 3.2 shows that Theorem 3.10 is a generalization of Theorem 2.2.

Corollary 3.11. *Let $\Omega \subset \mathbb{C}$ be a simply connected region and let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a piecewise-smooth path. Then $\text{Ind}_\gamma(z_0) = 0$ for all $z_0 \in \mathbb{C} \setminus \Omega$.*

Remark 3.14. Intuitively, this means that a simply connected region does not have holes in it.

Proof of Theorem 3.11. Given $z_0 \in \Omega \setminus \mathbb{C}$ the function

$$f(z) := \frac{1}{z - z_0}$$

is holomorphic on Ω . By Theorem 3.10 we have that

$$\text{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz = 0,$$

which completes the proof. $\square$

Remark 3.15. We can now show Example 3.4 by setting $\gamma: [0, 2\pi] \rightarrow \mathbb{C} \setminus \{0\}$ to be the positively oriented circle with center 0 and radius 1. Since $\text{Ind}_\gamma(0) = 1$, it follows that from Theorem 3.11 that $\mathbb{C} \setminus \{0\}$ is not simply connected.

3.4 The residue theorem

Remark 3.16. We now assume that paths are piecewise-smooth again.

Theorem 3.12 (The residue theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected region, and let f be meromorphic on Ω . Let A be the set of points in Ω at which f has a pole, and suppose that A is finite. Then for every closed path $\gamma: [a, b] \rightarrow \Omega \setminus A$,*

$$\int_\gamma f(z) dz = 2\pi i \sum_{z \in A} \text{Ind}_\gamma(z) \cdot \text{Res}(f, z).$$

Remark 3.17. The identity above is called the *residue formula*. It also holds in the case A is countable, in which only a finite amount of points have index different than 0, but the version above will be sufficient

Proof. Let $z_1, \dots, z_n$ be the distinct points of A . For each $1 \leq k \leq n$ denote m_k the order of the pole which f has at z_k , and define

$$Q_k(z) := \sum_{j=1}^{m_k} \frac{a_{k,j}}{(z - z_k)^j}$$

to be the principle part of f at the pole z_k . Note that $\text{Res}(f, z_k) = a_{k,1}$ by definition. Let $g: \Omega \setminus A \rightarrow \mathbb{C}$ be with

$$g(z) := f(z) - \sum_{k=1}^n Q_k(z)$$

for $z \in \Omega \setminus A$. Since g is holomorphic on $\Omega \setminus A$ from Theorem 3.4 there exists $\epsilon > 0$ so that $D_\epsilon(z_1) \subset \Omega$ and a holomorphic function $h: D_\epsilon(z_1) \rightarrow \mathbb{C}$ so that $h(z) = f(z) - Q_1(z)$ for all $z \in D_\epsilon(z_1)$. This means that $f - Q_1$ has a holomorphic extension to $D_\epsilon(z_1)$ so z_1 is a removable singularity. Similarly, all other singularities $z_2, \dots, z_n$ are removable so we may assume g is holomorphic on Ω .

Let $\gamma: [a, b] \rightarrow \Omega \setminus A$ be a closed path. Since g is holomorphic on Ω , since Ω is simply connected, and by Theorem 3.10 it follows that

$$\int_\gamma g(z)dz = 0 \implies \int_\gamma f(z)dz = \sum_{k=1}^n \int_\gamma Q_k(z)dz.$$

Let $1 \leq k \leq n$ be given. For each $j \geq 2$ the function $z \mapsto \frac{-1}{(j-1)(z-z_k)^{j-1}}$ is a primitive for the function $z \mapsto \frac{1}{(z-z_k)^j}$ on $\mathbb{C} \setminus \{z_k\}$. Thus, by Theorem 1.22 it follows that

$$\int_\gamma \frac{1}{(z-z_k)^j} dz = 0$$

for all $j \geq 2$, and from Theorem 1.27 it follows that

$$\int_\gamma \frac{1}{(z-z_k)^j} dz = 2\pi i \cdot \text{Ind}_\gamma(z_k).$$

This means that

$$\int_\gamma Q_k(z)dz = \sum_{j=1}^n a_{k,j} \int_\gamma \frac{1}{(z-z_k)^j} dz = 2\pi i \cdot \text{Ind}_\gamma(z_k) \cdot a_{k,1}.$$

Since we have seen earlier that $\text{Res}(f, z_k) = a_{k,1}$ we now get that

$$\int_\gamma f(z)dz = \sum_{k=1}^n \int_\gamma Q_k(z)dz = 2\pi i \sum_{k=1}^n \text{Ind}_\gamma(z_k) \cdot \text{Res}(f, z_k),$$

which completes the proof of the theorem. $\square$

Example 3.5. By using the residue formula we can show that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

This can be shown by considering $(\arctan)'(x) = (1+x^2)^{-1}$ for $x \in \mathbb{R}$, but we will still choose to show this using the residue formula.

The function

$$f(z) := \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

is meromorphic on $\mathbb{C}$ with poles at i and $-i$. Since the functions $z \mapsto (z \pm i)^{-1}$ is holomorphic and nonvanishing on a neighbourhood of $\pm i$ these are simple poles (this is because $(z \mp i)^{-1}(z \pm i)^{-1} = f(z)$ so the order is 1). It follows that

$$\text{Res}(f, i) = \lim_{z \rightarrow i} (z-i)f(z) = \frac{1}{2i}.$$

We can now consider the path γ_R for $R > 1$ which is the path travelling along $[-R, R]$ and then along $C_R(0)^+$ where

$$C_R(0)^+ := \{z \in C_R(0) \mid \text{Im}(z) \geq 0\}.$$

We can then verify that $\text{Ind}_{\gamma_R}(i) = 1$, and then by the residue theorem we get

$$\int_{\gamma_R} f(z) dz = 2\pi i \cdot \text{Ind}_{\gamma_R}(i) \cdot \text{Res}(f, i) = \pi.$$

Moreover, for $R > 2$ and $z \in C_R(0)^+$ we have

$$\left| 1 + z^2 \right| \geq \left| z^2 \right| - 1 \geq \frac{1}{2} |z|^2 = \frac{R^2}{2}.$$

It follows that

$$\left| \int_{C_R(0)^+} f(z) dz \right| \leq \text{length} \left(C_R(0)^+ \right) \cdot \|f\|_{C_R(0)^+} \leq \pi R \cdot \frac{2}{R^2} = \frac{2\pi}{R}.$$

It then follows that

$$\pi = \lim_{R \rightarrow \infty} \left(\int_{[-R, R]} f dz + \int_{C_R(0)^+} f dz \right) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

Example 3.6. Given $0 < a < 1$ be given. We can show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(\pi a)}.$$

Example 3.7. To be added.

Lemma 3.13 (Jordan's lemma). *Let f be a complex-valued, continuous function, defined on the semicircle*

$$C_R = \left\{ Re^{i\theta} \mid \theta \in [0, \pi] \right\}.$$

Suppose that $f(z) = e^{iaz}g(z)$ for all $z \in C_R$ for some $a > 0$. Then we have the bound

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} M_R$$

where $M_R := \max_{\theta \in [0, \pi]} |g(Re^{i\theta})|$. Moreover, equality holds if and only if $g(z)$ vanishes everywhere, in which case both sides are identically zero.

Remark 3.18. An analogous statement for a semicircle of the form

$$C_R = \left\{ Re^{i\theta} \mid \theta \in [\pi, 2\pi] \right\}.$$

when $a < 0$. When $a = 0$ we can't use Jordan's lemma but we may still be able to use the estimation lemma (Theorem 1.18).

Remark 3.19. If f satisfies the conditions of Jordan's lemma for all sufficiently large R and $\lim_{R \rightarrow \infty} M_R = 0$, then by the lemma

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

Definition 3.12 (Hyperbolic sine). We define the hyperbolic sine to be the odd part of the exponential function,

$$\sinh z := \frac{e^z - e^{-z}}{2} = \frac{e^{2z} - 1}{2e^z} = \frac{1 - e^{-2z}}{2e^{-z}}.$$

Definition 3.13 (Hyperbolic cosine). We define the hyperbolic cosine to be the odd part of the exponential function,

$$\cosh z := \frac{e^z + e^{-z}}{2} = \frac{e^{2z} + 1}{2e^z} = \frac{1 + e^{-2z}}{2e^{-z}}.$$

Definition 3.14 (Other hyperbolic functions). The rest of the hyperbolic functions are defined similarly to how they are defined in the real sense:

$$\begin{aligned} \tanh z &:= \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} \\ \coth z &:= \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{e^{2z} + 1}{e^{2z} - 1} \\ \operatorname{sech} z &:= \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}} = \frac{2e^z}{e^{2z} + 1} \\ \operatorname{csch} z &:= \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}} = \frac{2e^z}{e^{2z} - 1}. \end{aligned}$$

Example 3.8. To be added.

3.5 The argument principle and its applications

Theorem 3.14 (The argument principle). Let $\Omega \subset \mathbb{C}$ be open, let f be meromorphic on Ω , and let $z_0 \in \Omega$ and $r > 0$ be with $\overline{D}_r(z_0) \subset \Omega$. Suppose that f has no poles or zeros on $C_r(z_0)$. Denote:

$$\begin{aligned} Z_{D_r(z_0)} &= \text{number of zeros of } f \text{ in } D_r(z_0) \\ P_{D_r(z_0)} &= \text{number of poles of } f \text{ in } D_r(z_0). \end{aligned}$$

Then,

$$\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)} dz = Z_{D_r(z_0)} - P_{D_r(z_0)}.$$

Proof. We can elegantly see that there exists $R > r$ so that $\overline{D}_R(z_0) \subset \Omega$. Since f does not vanish on $C_r(z_0)$ it is not identically zero on $D_R(z_0)$. From this, and since f is meromorphic and $\overline{D}_R(z_0)$ is compact from Remark 3.8, it follows that f has at most finitely many zeros and finitely many poles in $D_R(z_0)$. This means that f'/f is meromorphic on $\overline{D}_R(z_0)$ with finitely many poles there, and without poles on $C_r(z_0)$. This means we can now calculate $\int_{C_r(z_0)} \frac{f'(z)}{f(z)} dz$ using the residue theorem.

Denote by A the set of zeros of f in $D_r(z_0)$, and by B the set of poles of f in $D_r(z_0)$. We also denote by m_w the order of a zero $w \in A$ of f , and by l_w the order of a pole $w \in B$. Note that the set of poles of f'/f in $D_r(z_0)$ is contained in $A \cup B$.

Let $w \in A$ be given, and denote $m := m_w$. Then by Theorem 3.1 we have that there exist $\epsilon > 0$ so that $D_\epsilon(w) \subset \Omega$ and a holomorphic $g: D_\epsilon(w) \rightarrow \mathbb{C}$ so that $g(z) \neq 0$ and $f(z) = (z - w)^m g(z)$ for each $z \in D_\epsilon(w)$. This implies that for $z \in D'_\epsilon(w)$ we have

$$\frac{f'(z)}{f(z)} = \frac{m(z - w)^{m-1}g(z) + (z - w)^m g'(z)}{(z - w)^m g(z)} = \frac{m}{z - w} + \frac{g'(z)}{g(z)}.$$

Since g'/g is holomorphic on $D_\epsilon(w)$, this equation shows that that f'/f has a simple pole at w , and that $\operatorname{Res}(f'/f, w) = m_w$.

Next Let $w \in B$ be given, and denote $l := l_w$. Then by Theorem 3.3 we have that there exist $\epsilon > 0$ so that $D_\epsilon(w) \subset \Omega$ and a holomorphic $h: D_\epsilon(w) \rightarrow \mathbb{C}$ so that $h(z) \neq 0$ $z \in D_\epsilon(w)$ and $f(z) = (z - w)^{-l} h(z)$ for each $z \in D'_\epsilon(w)$. This implies that for $z \in D'_\epsilon(w)$ we have

$$\frac{f'(z)}{f(z)} = \frac{-l(z - w)^{-l-1}h(z) + (z - w)^{-l}h'(z)}{(z - w)^{-l}h(z)} = \frac{-l}{z - w} + \frac{h'(z)}{h(z)}.$$

Since h'/h is holomorphic on $D_\epsilon(w)$, this equation shows that f'/f has a simple pole at w , and that $\text{Res}(f'/f, w) = -l_w$.

From the residue theorem (Theorem 3.12) we now have that

$$\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)} dz = \sum_{w \in A \cup B} \text{Res}(f'/f, w) = \sum_{w \in A} m_w - \sum_{w \in B} l_w,$$

which completes the proof of this theorem. $\square$

Remark 3.20. One way to remember if the integral is equal $Z_{D_r(z_0)} - P_{D_r(z_0)}$ or $P_{D_r(z_0)} - Z_{D_r(z_0)}$, it to remember it as the *ZP*-theorem.

Theorem 3.15 (Rouché's theorem). *Let $\Omega \subset \mathbb{C}$ be open, let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic, and let $z_0 \in \Omega$ and $r > 0$ be with $\overline{D_r(z_0)} \subset \Omega$. Suppose that $|f(z)| > |g(z)|$ for all $z \in C_r(z_0)$. Then f and $f + g$ have the same number of zeros in $D_r(z_0)$, where like before zeros are counted with their multiplicities.*

Proof. For $t \in [0, 1]$ we define $f_t: \Omega \rightarrow \mathbb{C}$ by

$$f_t(z) = f(z) + tg(z).$$

We denote the number of zeros of f_t in $D_r(z_0)$ by n_t . It is clear that f_t is holomorphic on Ω and that for $z \in C_r(z_0)$ we have

$$|f_t(z)| \geq |f(z)| - t|g(z)| > 0.$$

Thus, by the argument principle we have

$$n_t = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'_t(z)}{f_t(z)} dz.$$

It is also clear that the functions $(t, z) \mapsto f_t(z)$ and $(t, z) \mapsto f'_t(z)$ are continuous on $X := [0, 1] \times C_r(z_0)$, and since $f_t(z) \neq 0$ for $z \in X$, it follows that $(t, z) \mapsto \frac{f'_t(z)}{f_t(z)}$ is continuous on X . Moreover, since X is compact, it follows that $(t, z) \mapsto \frac{f'_t(z)}{f_t(z)}$ is uniformly continuous on X . This means that given a sequence $\{t_n\}_{n=1}^\infty \subset [0, 1]$ with $t_n \xrightarrow{n \rightarrow \infty} t_0$, the sequence $\{f'_{t_n}(z)/f_{t_n}(z)\}_{n=1}^\infty$ converges to $\frac{f'_{t_0}(z)}{f_{t_0}(z)}$ uniformly on $C_r(z_0)$ as $n \rightarrow \infty$. According to Theorem 2.5 and by the formula for n_t it follows that the function $t \rightarrow n_t$ is continuous on $[0, 1]$. Since $n_t \in \mathbb{N}$ for all $t \in [0, 1]$ it follows that n_t is constant, and in particular $n_0 = n_1$ which completes the proof. $\square$

Remark 3.21. Here is some intuition for Rouché's theorem: We know by Remark 2.4 that the values of holomorphic functions on $D_r(z_0)$ is determined by their values on $C_r(z_0)$. Therefore, if $h = f + g$ is a function so that $|g(z)| \leq |f(z)|$ on $C_r(z_0)$, and intuitively f is a nice function, and g is a perturbation, then the inequality $|g(z)| \leq |f(z)|$, tells us that the perturbation is small enough, so that h (the original function) and $h - g = f$ (the nice function, or equivalently, the original function without the perturbation), have the same amount of zeros (counting multiplicities). Another approach, is to consider $h = f + g$ some given function and let $-g$ be the perturbation. Then if $|-g(z)| \leq |h(z)|$ on $C_r(z_0)$ then h and $h + (-g) = f$ have the same amount of zeros. We usually choose $-g$ so that $h + (-g) = h - g = f$ is a nice function.

Exercise 3.1. Find the number of zeros, counting multiplicities, of the polynomial $p(z) = 2z^8 + 3z^5 - 9z^3 + 2$ in the annulus $1 \leq |z| < 2$.

Solution. First we notice that the solution is the amount of zeros of $p(z)$ inside $D_2(0)$ minus the amount of zeros of $p(z)$ inside $D_1(0)$. In order to find the number of zeros of $p(z)$ in each case, we need to think which part of p is the “nice” part, and which is the “perturbating” part. We need to be able to find the zeros of the “nice” part so it’s most likely just a single term.

On $C_2(0)$, since $|z| = 2$, it is clear that the term $f(z) = 2z^8$ is the dominant part of the function, so by defining if we denote $q(z) := -g(z) = -3z^5 + 9z^3 - 2$ as our perturbation, since p and q are entire, and since it is clear that $|p(z)| > |q(z)|$ for $z \in C_2(0)$, by Rouché’s theorem it follows that the number of zeros of $p(z)$ and $(p+q)(z) = f(z) = 2z^8$ in $D_2(0)$ is the same, so $p(z)$ has exactly 8 zeros (counting multiplicities) in $D_2(0)$.

On the other hand, if we define $g(z) := -9z^3$ and $\tilde{p}(z) := 2z^8 + 3z^5 + 2$. We see that $|g(z)| > |\tilde{p}(z)|$ on the unit circle $C_1(0)$, which means that g and $g + \tilde{p} = p$ have the same number of roots in the unit disc, so p has exactly 3 zeros (counting multiplicities) in the unit disc.

Using the previous calculations it is clear that the number of zeros of p in the annulus $1 \leq |z| < 2$ counting multiplicities is $8 - 3 = 5$.

Definition 3.15 (Open map). Let $\Omega \subset \mathbb{C}$ be open. A map $f: \Omega \rightarrow \mathbb{C}$ is said to be open if $f(U)$ is open for every open $U \subset \Omega$.

Theorem 3.16 (Open mapping theorem). Let Ω be a region, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic and nonconstant. Then f is an open map.

Proof. Since f is nonconstant, it follows by Theorem 2.13 that the set $f^{-1}\{w_0\}$ has no limit points in Ω (sets like these are called discrete sets). Since $\mathbb{C}$ is separable (which means it has a countable dense subset) it follows that $f^{-1}\{w_0\}$ is at most countable. Thus exists $\delta > 0$ so that $\overline{D}_\delta(z_0) \subset U$ and $f(z) \neq w_0$ for all $z \in C_\delta(z_0)$. Since $C_\delta(z_0)$ is compact and f is continuous, the function $z \mapsto f(z) - w_0$ attains a minimum on $C_\delta(z_0)$, so there exists $\epsilon > 0$ so that $|f(z) - w_0| > \epsilon$ for all $z \in C_\delta(z_0)$.

Let $w \in D_\epsilon(w_0)$ be given, and define $F, G: U \rightarrow \mathbb{C}$ by $F(z) = f(z) - w_0$ and $G(z) = w_0 - w$ (where G is constant). For $z \in C_\delta(z_0)$ we have,

$$|F(z)| = |f(z) - w_0| > \epsilon > |w_0 - w| = |G(z)|.$$

Thus, by Rouché theorem (Theorem 3.15), the functions F and $F + G$ have the same number of zeros in $D_\delta(z_0)$. In particular, since $F(z_0) = 0$, there exists $z \in D_\delta(z_0) \subset U$ so that

$$0 = F(z) + G(z) = f(z) - w,$$

which implies that $f(z) = w$ so $w \in f(U)$. Thus $D_\epsilon(w_0) \subset f(U)$, which completes the proof of the theorem. Let U be a nonempty open subset of Ω , let $z_0 \in U$ and denote $w_0 := f(z_0)$. In order to prove this theorem it suffices to show that there exists $\epsilon > 0$ so that $D_\epsilon(w_0) \subset f(U)$. $\square$

Theorem 3.17 (Maximum modulus principle). Let Ω be a region, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic and nonconstant. Then f does not attain a maximum in Ω .

Proof. Let $z_0 \in \Omega$. We will show that f does not attain a maximum at z_0 which completes the proof. Since f is nonconstant we can assume that $f(z_0) \neq 0$, and from the open mapping theorem (Theorem 3.16) we get that $f(\Omega)$ is open, so there exists $\epsilon > 0$ so that $D_\epsilon(f(z_0)) \subset f(\Omega)$. In particular this means that exists $t > 0$ so that $(1+t)f(z_0) \in f(\Omega)$. Let $z \in \Omega$ be with $f(z) = (1+t)f(z_0)$. We now get that

$$|f(z)| = (1+t)|f(z_0)| > |f(z_0)|,$$

which completes the proof of the theorem. $\square$

Corollary 3.18. *Let $\Omega \subset \mathbb{C}$ be a region and suppose that $\overline{\Omega}$ is compact. Let $f: \overline{\Omega} \rightarrow \mathbb{C}$ be continuous on $\overline{\Omega}$ and holomorphic on Ω . Then*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \partial\Omega} |f(z)|.$$

Proof. Assuming that f is constant makes this corollary trivial so we may assume it is non-constant. Since f is continuous on the compact set $\overline{\Omega}$ it attains a maximum at some $z_0 \in \overline{\Omega}$. From the maximum modulus principle it follows that $z_0 \notin \Omega$ which implies that $z_0 \in \partial\Omega$, which completes the proof. $\square$

3.6 The complex logarithm

Remark 3.22. Suppose we wanted to define an inverse for the function $z \mapsto e^z$. Intuitively, if $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$, we would like to set $\log z := \log r + i\theta$, where $\log r$ is the standard logarithm for the real number r . However, the problem with that is that θ is not uniquely determined. In this subsection, we will show how to define this function properly.

Proposition 3.19. *Let $\Omega \subset \mathbb{C}$ be a simply connected region, let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic, and suppose that $f(z) \neq 0$ for all $z \in \Omega$. Then there exists a holomorphic $g: \Omega \rightarrow \mathbb{C}$ so that $f(z) = e^{g(z)}$ for all $z \in \Omega$*

Corollary 3.20. *Let $\Omega \subset \mathbb{C}$ be a simply connected region with $0 \notin \Omega$ and $1 \in \Omega$, and let $\epsilon > 0$ be with $D_\epsilon(1) \subset \Omega$. Then there exists a unique holomorphic function $g: \Omega \rightarrow \mathbb{C}$ so that $e^{g(z)} = z$ for $z \in \Omega$ and $g(r) = \log r$ for $r \in (1 - \epsilon, 1 + \epsilon)$, where $\log r$ is the standard real logarithm. Moreover, $g'(z) = 1/z$ for each $z \in \Omega$.*

Remark 3.23. The function g is called the branch of the logarithm corresponding to Ω , and is sometimes denoted by $\log_\Omega$.

Proof of Theorem 3.20. Since Ω is simply connected and $0 \notin \Omega$, by Theorem 3.19 there exists a holomorphic $\tilde{g}: \Omega \rightarrow \mathbb{C}$ so that $z = e^{\tilde{g}(z)}$ for all $z \in \Omega$. From $1 = e^{\tilde{g}(1)}$ it follows that $\tilde{g}(1) = 2\pi in$ for some $n \in \mathbb{Z}$. For $z \in \Omega$ set $g(z) := \tilde{g}(z) - 2\pi in$. It is clear that g is holomorphic on Ω and $g(1) = 0$. Since the function $\exp$ is $2\pi i$ -periodic it follows that $e^{g(z)} = z$ for $z \in \Omega$. This gives that for $z \in \Omega$

$$1 = (\exp \circ g)'(z) = g'(z)e^{g(z)} = g'(z)z,$$

which gives $g'(z) = 1/z$. Thus, for $r \in (1 - \epsilon, 1 + \epsilon)$

$$g(r) = g(r) - g(1) = \int_{[1,r]} g'(z) dz = \int_{[1,r]} \frac{1}{z} dz = \int_1^r \frac{1}{t} dt = \log r.$$

The uniqueness of g follows from Theorem 1.24, which completes the proof. $\square$

Example 3.9. In Example 3.3 we saw that $\Omega := \mathbb{C} \setminus (-\infty, 0]$ is simply connected. In this case, we denote the function $\log_\Omega$ by $\log$, and it is called the principle branch of the logarithm. Given $r > 0$ and $\theta \in (-\pi, \pi)$, let us show that $\log(re^{i\theta}) = \log r + i\theta$. Indeed, if γ is the path travelling along $[1, r]$, and then along the arc η from r to $z := re^{i\theta}$, then

$$\log z = \int_\gamma \frac{1}{w} dw = \int_{[1,r]} \frac{dw}{w} + \int_\eta \frac{dw}{w} = \int_0^r \frac{1}{t} dt + \int_0^\theta \frac{ire^{it}}{re^{it}} dt = \log r + i\theta,$$

which completes the example.

Remark 3.24. As opposed to the real case, it is not true in general that $\log(z_1 z_2) = \log z_1 + \log z_2$. Indeed, for $z_1 = z_2 = e^{2\pi i/3}$ we have

$$\log(z_1 z_2) = \log(e^{-2\pi i/3}) = -i\frac{2\pi}{3} \neq i\frac{4\pi}{3} = \log z_1 + \log z_2.$$

Definition 3.16 (Complex powers). Let $\Omega \subset \mathbb{C}$ be a simply connected region with $0 \notin \Omega$ and $1 \in \Omega$, and let $\log_\Omega$ be the branch of the logarithm corresponding to Ω . Given $\alpha \in \mathbb{C}$ we define $z^\alpha := e^{\alpha \log_\Omega z}$ for each $z \in \Omega$.

Remark 3.25. We have $1^\alpha = 1$ and that for each $n \geq 1$ and $z \in \Omega$

$$\left(z^{1/n}\right)^n = \left(e^{\frac{1}{n} \log_\Omega z}\right)^n = z.$$

4 Conformal mappings

4.1 Conformal mappings and conformal equivalence

Definition 4.1 (Conformal map). Let $\Omega \subset \mathbb{C}$ be open. A function $f: \Omega \rightarrow \mathbb{C}$ is said to be conformal if it is holomorphic on Ω and $f'(z) \neq 0$ for all $z \in \Omega$.

Theorem 4.1. Let $\Omega \subset \mathbb{C}$ be open, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic and injective. Then f and $f^{-1}: f(\Omega) \rightarrow \Omega$ are both conformal.

Remark 4.1. By the open mapping theorem it follows that $f(\Omega)$ is open.

Proof. To be added. □

Definition 4.2 (Conformal equivalency). Let U and V be open subsets of $\mathbb{C}$. We say that U and V are conformally equivalent if there exists a holomorphic bijection $f: U \rightarrow V$.

Remark 4.2. By Theorem 4.1, it follows that U and V are conformally equivalent if and only if there exists a conformal bijection $f: U \rightarrow V$ with $f^{-1}: V \rightarrow U$ also being conformal.

Definition 4.3 (Upper half-plane). The upper half-plane is the set

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

Example 4.1. Denote by $\mathbb{D}$ the open disc $D_1(0)$, and by $\mathbb{H}$ the upper half-plane. Let us show that $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent.

Let us define the functions $F: \mathbb{H} \rightarrow \mathbb{C}$ and $G: \mathbb{D} \rightarrow \mathbb{C}$ with

$$\begin{aligned} F(z) &:= \frac{i - z}{i + z} \\ G(z) &:= i \frac{1 - z}{1 + z}. \end{aligned}$$

It is clear that F and G are both holomorphic on their respective domains. Since any $z \in \mathbb{H}$ is closer to i than $-i$ and thus $F(z) < 1$, which implies that $F(\mathbb{H}) \subset \mathbb{D}$. Let $x + iy = z \in \mathbb{D}$ with $x, y \in \mathbb{R}$ be given. Then,

$$\operatorname{Im}(G(z)) = \operatorname{Re}\left(\frac{1 - x - iy}{1 + x + iy}\right) = \frac{1 - x^2 - y^2}{(1 + x)^2 + y^2} > 0,$$

which shows that $G(\mathbb{D}) \subset \mathbb{H}$. For $z \in \mathbb{D}$ we have

$$F(G(z)) = F\left(i \frac{1 - z}{1 + z}\right) = \frac{i - i \frac{1 - z}{1 + z}}{i + i \frac{1 - z}{1 + z}} = z,$$

and similarly $G(F(z)) = z$ for $z \in \mathbb{H}$. This shows that $F: \mathbb{H} \rightarrow \mathbb{D}$ is a holomorphic bijection, so by definition $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent.

Remark 4.3. Mappings of the form $\varphi(z) := \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ and $ad - cb \neq 0$, are called Möbius transformations (or fractional linear transformations). By introducing a new point ∞ (called the point at infinity) and setting $\varphi(-d/c) := \infty$ and $\varphi(\infty) := a/c$ if $c \neq 0$ and $\varphi(\infty) := \infty$ if $c = 0$, these maps become bijections from $\mathbb{C} \cup \{\infty\}$ into itself. Note that these definitions imply the conventions $z/\infty = 0$, $z + \infty = \infty$ and $w/0 = \infty$ for all $z \in \mathbb{C}$ and $w \in \{0\}$.

4.2 The Schwartz lemma and automorphisms of the disc

Lemma 4.2 (The Schwartz lemma). *Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,*

- (1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$;
- (2) $|f'(0)| \leq 1$;
- (3) if for some $0 \neq z_0 \in \mathbb{D}$ we have $|f(z_0)| = |z_0|$ then f is a rotation;
- (4) if $|f'(0)| = 1$ then f is a rotation.

Proof.

- (1) Let $g: \mathbb{D} \setminus \{0\} \rightarrow \mathbb{C}$ be with $g(z) := f(z)/z$. Since $f(0) = 0$, it follows from Theorem 3.1 that g has a removable singularity at 0, so we may assume that it is defined and holomorphic on $\mathbb{D}$. By Theorem 3.18 and since $f(\mathbb{D}) \subset \mathbb{D}$, it follows that for all $0 \neq z \in \mathbb{D}$ and $|z| < r < 1$

$$\frac{|f(z)|}{|z|} = |g(z)| \leq \sup_{w \in C_r(0)} |g(w)| = \sup_{w \in C_r(0)} \frac{|f(w)|}{|w|} \leq r^{-1}.$$

By letting $r \rightarrow 0$ we obtain that $|f(z)| \leq |z|$ which completes the first part of the lemma.

- (2) The result from (1) also gives us

$$|f'(0)| = \lim_{z \rightarrow 0} \left| \frac{f(z)}{z} \right| \leq 1,$$

which is exactly what we wanted to show.

- (3) Suppose that there exists $0 \neq z_0 \in \mathbb{D}$ such that $|f(z_0)| = |z_0|$. This implies that $|z_0| = 1$. However, by part (1), this implies that g attains a maximum at z_0 . Thus by the maximum modulus principle, it follows that g is constant, so there exists $c \in \mathbb{C}$ such that $g(z) = c$ for all $z \in \mathbb{D}$, such that $|c| = |g(z_0)| = 1$. By the definition of g , we obtain that f is a rotation, which completes the proof for the third part.

- (4) Suppose that $|f'(0)| = 1$. By the continuity of g at 0 we obtain

$$1 = |f'(0)| = \lim_{z \rightarrow 0} \left| \frac{f(z)}{z} \right| = \lim_{z \rightarrow 0} |g(z)| = |g(0)|.$$

This means that g attains a maximum at 0, and thus by the maximum modulus principle, from the same argument in (3) we get that f is a rotation, which completes the last part of the lemma.

□

Definition 4.4 (Automorphism). Let $\Omega \subset \mathbb{C}$ be open and nonempty. A holomorphic bijection $f: \Omega \rightarrow \Omega$ is called an automorphism of Ω . The set of all automorphisms on Ω is denoted by $\text{Aut}(\Omega)$. It is a group with respect to composition, where the identity of the group is the identity map $z \mapsto z$.

Remark 4.4. By Theorem 4.1 it follows that f and f^{-1} are both conformal.

Example 4.2. For every $\theta \in \mathbb{R}$ the rotation $R_\theta(z) := e^{i\theta}z$ is an automorphism of $\mathbb{D}$. Its inverse is $R_{-\theta}(z)$.

Remark 4.5. It's good to understand the following example intuitively.

Example 4.3. For $\alpha \in \mathbb{D}$ we define ψ_α as the linear fractional transformation

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Since the only pole of ψ_α is $z = 1/\bar{\alpha} \notin \mathbb{D}$, it is holomorphic. Given $z \in C_1(0)$ we have $z\bar{z} = 1$, so by setting $w := 1 - \bar{\alpha}z$ we obtain

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z} = \frac{z(\alpha\bar{z} - 1)}{1 - \bar{\alpha}z} = -z \frac{\bar{w}}{w},$$

which implies that $|\psi_\alpha(z)| = 1$. Thus by Theorem 3.18 we have that $\psi_\alpha(\mathbb{D}) \subset \mathbb{D}$. Moreover, for $z \in \mathbb{D}$ we have

$$\psi_\alpha \circ \psi_\alpha(z) = \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha}z}}{1 - \bar{\alpha} \frac{\alpha - z}{1 - \bar{\alpha}z}} = \frac{\alpha(1 - \bar{\alpha}z) - \alpha + z}{1 - \bar{\alpha}z - \bar{\alpha}(\alpha - z)} = \frac{-\alpha\bar{\alpha}z + z}{1 - \bar{\alpha}\alpha} = z,$$

which implies that $\psi_\alpha \in \text{Aut}(\mathbb{D})$, and $\psi_\alpha^{-1} = \psi_\alpha$.

Remark 4.6. We have that $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$. The automorphism ψ_α interchanges between 0 and α .

Proposition 4.3. Let $f \in \text{Aut}(\mathbb{D})$ be given. Then there exists $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$ so that $f = R_\theta \circ \psi_\alpha$.

Proof. First set $\alpha = f^{-1}(0)$ and $g := f \circ \psi_\alpha$. It follows that $g \in \text{Aut}(D)$ and also $g(0) = 0$. We can now apply the Schwartz lemma to g^{-1} and get that $|g^{-1}(z)| \leq |z|$ for $z \in D$, which implies that $|z| \leq |g(z)|$ for $z \in D$. We can apply a similar argument using the Schwartz lemma on g and get that $|g(z)| = |z|$ for $z \in D$ which means that $g = R_\theta$ for some $\theta \in \mathbb{R}$. This means that $f = g \circ \psi_\alpha^{-1} = R_\theta \circ \psi_\alpha$, which completes the proof. $\square$

Corollary 4.4. The only automorphisms of $\mathbb{D}$ that fix the origin are the rotations.

Proof. Let $f \in \text{Aut}(D)$ be with $f(0) = 0$. By Theorem 4.3 there exist $\alpha \in D$ and $\theta \in \mathbb{R}$ so that $f = R_\theta \circ \psi_\alpha$. Hence we have

$$0 = f(0) = R_\theta(\psi_\alpha) = e^{i\theta}\alpha,$$

which implies that $\alpha = 0$. Since $\psi_0 = R_\pi$ we obtain that $f = R_\theta \circ R_\pi = R_{\theta+\pi}$, which completes the proof. $\square$

4.3 The Riemann mapping theorem

Remark 4.7. We want to answer the following question: Given an open subset $\Omega \subset \mathbb{C}$, what are the necessary and sufficient conditions under which Ω is conformally equivalent to $\mathbb{D}$.

We know that if Ω and $\mathbb{D}$ are conformal, they are also homeomorphic, and since simple connectivity is a topological property, and the disc is simply connected, Ω must also be simply connected.

We also have that $\Omega \neq \mathbb{C}$ since otherwise by Liouville's theorem f would be constant.

The Riemann mapping theorem states that these necessary conditions, are also sufficient.

Theorem 4.5 (Riemann mapping theorem). Let $\Omega \subset \mathbb{C}$ be a simply connected region with $\Omega \neq \mathbb{C}$. Then for each $z_0 \in \Omega$ there exists a unique holomorphic bijection $f: \Omega \rightarrow \mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.

Corollary 4.6. Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}$. Suppose that $\Omega_i \neq \emptyset, \mathbb{C}$ and that Ω_i is simply connected for $i = 1, 2$. Then Ω_1 and Ω_2 are conformally equivalent.

Proof. By the Riemann mapping theorem there exist holomorphic bijections $f_i: \Omega_i \rightarrow \mathbb{D}$ for $i \in \{1, 2\}$. Since $f_2^{-1} \circ f_1: \Omega_1 \rightarrow \Omega_2$ is a holomorphic bijection, it follows that Ω_1 and Ω_2 are conformally equivalent. $\square$

Remark 4.8. For the proof of this theorem we need to lay out some preparatory definitions and theorems.

4.3.1 Arzelá–Ascoli theorem

Definition 4.5 (Equicontinuity). Let $K \subset \mathbb{C}$ be compact and let $\mathcal{F} \subset C(K)$. We say that $\mathcal{F}$ is equicontinuous if for every $\epsilon > 0$ there exist $\delta > 0$ so that $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$ and $z, w \in K$ with $|z - w| < \delta$.

Definition 4.6 (Uniform boundedness). Let $K \subset \mathbb{C}$ be compact and let $\mathcal{F} \subset C(K)$. We say that $\mathcal{F}$ is uniformly bounded if there exists $C > 1$ so that $\|f\|_K \leq C$ for all $f \in \mathcal{F}$.

Theorem 4.7 (Arzelá–Ascoli theorem). *Let $K \subset \mathbb{C}$ be compact, and let $\mathcal{F} \subset C(K)$ be given. Suppose that $\mathcal{F}$ is equicontinuous and uniforml bounded. Then for every sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ and $f \in C(K)$ so that $f_{n_k} \xrightarrow{k \rightarrow \infty} f$ uniformly on K .*

Proof. To be added $\square$

4.3.2 Montel’s theorem

Definition 4.7 (Normal family). Let $\Omega \subset \mathbb{C}$ be open, and let $\mathcal{F}$ be a family of holomorphic functions on Ω . We say that $\mathcal{F}$ is a normal family if for every sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ and a function $f: \Omega \rightarrow \mathbb{C}$ so that $f_{n_k} \xrightarrow{k \rightarrow \infty} f$ uniformly on compact subsets of Ω .

Definition 4.8 (Uniform boundedness on compact subsets). Let $\Omega \subset \mathbb{C}$ be open, and let $\mathcal{F}$ be a family of holomorphic functions on Ω . We say that $\mathcal{F}$ is uniformly bounded on compact subsets of Ω if for every compact $K \subset \Omega$ there exists $C_K > 1$ so that $\|f\|_K \leq C_K$ for all $f \in \mathcal{F}$.

Theorem 4.8 (Montel’s theorem). *Let $\Omega \subset \mathbb{C}$ be open, let $\mathcal{F}$ be a family of holomorphic functions on Ω , and suppose that $\mathcal{F}$ is uniformly bounded on compact subsets of Ω . Then $\mathcal{F}$ is a normal family.*

Proof. $\square$

Definition 4.9 (Exhaustion). Let $\Omega \subset \mathbb{C}$ be open. A sequence $\{K_n\}_{n=1}^\infty$ of compact subsets of Ω is called an exhaustion of Ω if $K_n \subset \text{Int}(K_{n+1})$ for all $n \geq 1$ and $\Omega = \bigcup K_n$.

Remark 4.9. We can show that if $\{K_n\}_{n=1}^\infty$ is an exhaustion of Ω , then for every compact $K \subset \Omega$ there exists $n \geq 1$ so that $K \subset K_n$.

Remark 4.10. Every open $\Omega \subset \mathbb{C}$ has an exhaustion, since for each $n \geq 1$ we can set

$$K_n := \left\{ z \in \overline{D}_n(0) : d(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n} \right\},$$

and then $\{K_n\}_{n=1}^\infty$ is an exhaustion of Ω .

4.3.3 Additional preparation

Proposition 4.9. *Let $\Omega \subset \mathbb{C}$ be a region, and let $f, f_1, f_2, \dots, : \Omega \rightarrow \mathbb{C}$ be holomorphic. Suppose that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on compact subsets of Ω , and that f_n is injective for every $n \geq 1$. Then f is either injective or constant.*

Proof. Assume by contradiction that f is neither injective nor constant. Then there exist distinct $z_1, z_2 \in \mathbb{C}$ so that $f(z_1) = f(z_2)$ and the function $g := f - f(z_1)$ is not constant. We also define the functions $g_n := f_n - f(z_1)$. Since Ω is a region, by Theorem 2.12 there exists $r > 0$ so that $\overline{D}_r(z_2) \subset \Omega \setminus \{z_1\}$ and $g(z) \neq 0$ for $z \in C_r(z_2)$ or $z \in D_r(z_2)$. Since $g(z_2) = 0$ the argument principle gives us that

$$1 = \frac{1}{2\pi i} \int_{C_r(z_2)} \frac{g'(z)}{g(z)} dz \in \mathbb{Z}_+.$$

For each $n \geq 1$ we have f_n is injective so $g_n(z) \neq g_n(z_1) = 0$ for all $z \in \Omega \setminus \{z_1\}$, thus from $z \notin \overline{D}_r(z_2)$ we get from the argument principle that

$$0 = \frac{1}{2\pi i} \int_{C_r(z_2)} \frac{g'_n(z)}{g_n(z)} dz \text{ for all } n \geq 1.$$

Since $g_n \xrightarrow{n \rightarrow \infty} g$ uniformly on compact subsets of Ω , it follows that $g'_n \xrightarrow{n \rightarrow \infty} g'$ uniformly on compact subsets of Ω . From this and since g does not vanish on $C_r(z_2)$ it follows that $g'_n \xrightarrow{n \rightarrow \infty} g'$ uniformly on $C_r(z_2)$. Thus, by Theorem 2.5 we have that

$$0 = \lim_{n \rightarrow \infty} \int_{C_r(z_2)} \frac{g'_n(z)}{g_n(z)} dz = \int_{C_r(z_2)} \frac{g'(z)}{g(z)} dz \in \mathbb{Z}_+$$

and this contradiction completes the proof of the proposition. $\square$

4.3.4 Proof of the Riemann mapping theorem

Remark 4.11. Recall that the statement of the theorem is

Theorem 4.10 (Riemann mapping theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected region with $\Omega \neq \mathbb{C}$. Then for each $z_0 \in \Omega$ there exists a unique holomorphic bijection $f: \Omega \rightarrow \mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. To be added $\square$

5 Additional topics

5.1 Hurwitz's theorem

5.2 Picard's theorem

5.3 Laurent series