Algebraic Topology

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Based on lectures by Nir Lazarovich

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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1 Homology

1.1 Δ-complexes

In algebraic topology we study maps that take topological spaces to algebraic data.

$$\{\text{Top spaces}\} \to \text{Algebraic data}$$

one instance of these maps is the map $\pi_0(X)$ that we saw in point set topology

$$X \mapsto \pi_0(X) = \{\text{path connected components}\}$$

another instance it the map $\pi_1(X)$ that we say in topology

$$X, x_0 \mapsto \pi_1(X, x_0) = \{\text{the fundamental group of } X \text{ based at } x_0\}.$$

Remark 1.1. In general elements of π_n are maps from S^n to the space.

Definition 1.1 (Affinitely independent vectors)**.** Let $v_0, \dots, v_n \in \mathbb{R}^n$. They are called affinitely independent if $v_1 - v_0, \dots, v_n - v_0$ are linearly independent.

Definition 1.2 (n-simplex)**.** An n-simplex is the convex hull of $n+1$ affinitely independent points in $\mathbb{R}^m$.

Example 1.1. A 0-simplex is a point. A 1-simplex is an edge. A 2-simplex is a triangle. A 3-simplex is a tetraeder.

Definition 1.3 (Standard n-simplex)**.** The standard n-simplex is given by the convex hull of the $n+1$ unit vectors in $\mathbb{R}^{n+1}$

$$\Delta^n := [e_0, \dots, e_N].$$

Remark 1.2. We see that Δ^n parameterizes all other n-simplexes, with the map

$$(t_0, t_1, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

Definition 1.4 (Barycentric map)**.** The map $b_{[v_0,\dots,v_n]} \colon \Delta^n \mapsto [v_0, \dots, v_n]$ is called the barycentric map. It is given by

$$b_{[v_0,\dots,v_n]}(t_0, t_1, \dots, t_n) = \sum_{i=0}^{n} t_i v_i.$$

Definition 1.5 (Face)**.** A face of $[v_1, \dots, v_n]$ is given by

$$[v_1, \dots, \hat{v}_i, \dots, v_n] = [v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

for any $1 \leq i \leq n$.

Remark 1.3. The face of $[v_0, \dots, v_n]$ is an $(n-1)$-simplex. The ordering of the face is induced by the ordering of the original n-simplex.

Definition 1.6 (Boundary of a simplex)**.** Let Δ^n be a simplex. The boundary of Δ^n is defined as $\partial\Delta^n = \bigcup\{\text{faces of } \Delta^n\}$. Moreover, we now have the interior of the simplex which is $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$.

Definition 1.7 (Δ-complex)**.** A Δ-complex structure on a topological space X is a collection of continuous maps $\{\sigma_\alpha \colon \Delta^{n_\alpha} \to X\}_\alpha$ so that

(1) $\sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}}$ is injective and each $x \in X$ is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}}$.

(2) The restriction of σ_α to a face of Δ^{n_α} is some σ_β.

(3) $U \subset X$ is open if and only if $\sigma_\alpha^{-1}(U) \subset \Delta^{n_\alpha}$ is open for all α.

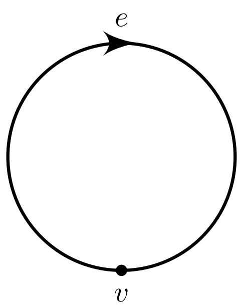
Remark 1.4. In (2) from Definition 1.7 make sure you understand that

$$\sigma_\beta = \sigma_\alpha \circ b_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \colon \Delta^{n-1} \to X,$$

which is equivalent to $\sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$.

Remark 1.5. We sometimes use the notation Δ_n to represent the set of all simplices of dimension n. The set Δ_0 represents all the vertices in the Δ-complex, the set Δ_1 represents all the edges etc.

Example 1.2 (Δ-complex of S^1)**.** Let S^1 the unit circle in $\mathbb{R}^2$, we define the Δ-complex:



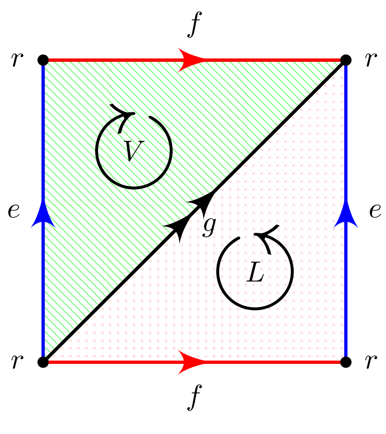
with the maps

$$v\colon \Delta^0 \to S^1$$
$$e\colon \Delta^1 \to S^1$$

where v maps its point Δ^0 to the vertex v and e maps its interval $\Delta^1 = [v_0, v_1]$ to the edge e.

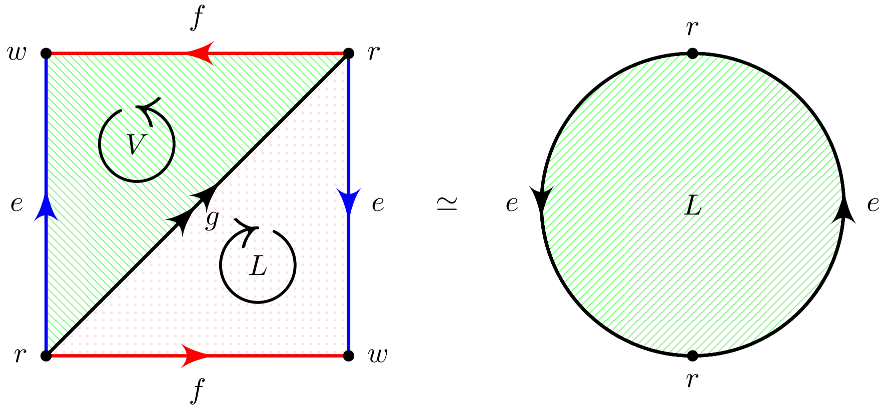
Remark 1.6. A nonexample of constructing S^1 is defining $\aleph$ different Δ^0 simplicies for each point of S^1. The reason this is a nonexample, is because condition (3) from Definition 1.7 would imply any subset of S^1 is open. Although this pathological structure is a nonexample of S^1 with its standard topology, it is a completely valid Δ-simplex of S^1 with the discrete topology.

Example 1.3 (Δ-complex of T^1)**.** Let $T = S^1 \times S^1$ be the standard torus.



This Δ-complex of the torus is the collection of $r\colon \Delta^0 \to T$ and $e, f, g\colon \Delta^1 \to T$ and $V, L\colon \Delta^2 \to T$ as can be seen in the diagram.

Example 1.4 (Δ-complex of $\mathbb{RP}^2$)**.** Let $\mathbb{RP}^2 := \mathbb{D}^2/\forall x \in S^1 : x \sim -x$ be the following space.



This Δ-complex of the real projective plane is the collection of $r, w\colon \Delta^0 \to \mathbb{RP}^2$ and $e, f, g\colon \Delta^1 \to \mathbb{RP}^2$ and $V, L\colon \Delta^2 \to \mathbb{RP}^2$ as can be seen in the diagram. Notice that the right hand side is not a Δ-complex, it is just the quotient space that represents the real projective plane.

Remark 1.7 (Orienatation of the faces)**.** In Example 1.4, we need to have all of the edges in the faces V, L, in a very specific order. For example if $V\colon [v_0, v_1, v_2] \to \mathbb{RP}^2$ then by part (2) of Definition 1.7 the maps $V|_{[v_0,v_1]}$, $V|_{[v_1,v_2]}$ and $V|_{[v_0,v_2]}$ must all be part of the sigma complex. The circular arrows represent if the vertices v_0, v_1, and v_2 are order clockwise or counterclockwise. In this case the orientation of V is counterclockwise and indeed $V|_{[v_0,v_1]} = g$, $V|_{[v_1,v_2]} = f$, and $V|_{[v_0,v_2]} = e$ so all the edges are in the Δ-complex. If this weren't the case then this would not have been a Δ-complex.

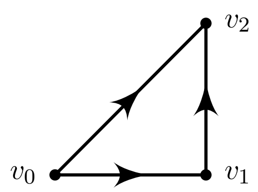
Remark 1.8. The space X is determined by the data of the Δ-complex. This means that the CW-complex of the generated by the Δ-complex is homeomorphic to X.

Definition 1.8 (Simplicial complex)**.** A simplicial complex is a Δ-complex so that each simplex $\sigma_\alpha\colon \Delta^n \to X$ is

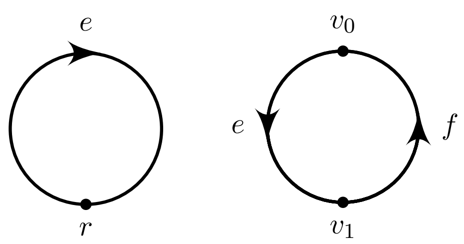
(1) injective (it suffices to check that it is injective on the vertices).

(2) σ_α is determined by its vertices.

Example 1.5. One of the simplest examples is the triangle.



Example 1.6. The Δ-complexes below are *nonexamples* of simplicial complexes.



The Δ-complex on the left does not satisfy condition (1) in Definition 1.8 since e is not injective. The Δ-complex on the right does not satisfy condition (2) in Definition 1.8 since both e and f are not determined by their vertices.

Remark 1.9. A simplicial complex is a generalization of simple graphes.

Remark 1.10. We can always cover any space by just using a vertex for any point in X, but this will be a Δ-complex of X endowed with the discrete topology.

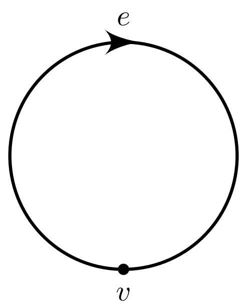
1.2 Simplicial homology

Definition 1.9 (Chains)**.** Let X be a topological space with Δ-structure $\{\sigma_\alpha^{n_\alpha}\colon \Delta^{n_\alpha} \to X\}_\alpha$. Then for any $n \geq 1$ we define the group $C_n^\Delta(X)$ in one of the following ways:

$$\begin{aligned} C_n^\Delta(X) &:= \text{free abelian group with basis } \{\sigma_\alpha^n\}_\alpha \\ &:= \bigoplus \mathbb{Z}\sigma_\alpha^n \\ &:= \operatorname{span}_\mathbb{Z}\{\sigma_\alpha^n\} \\ &:= \left\{\sum c_\alpha \sigma_\alpha^n \middle| c_\alpha \in \mathbb{Z} \text{ all but finitely many are } 0\right\}. \end{aligned}$$

Remark 1.11. An element of $C_n^\Delta(X)$ is called an n-chain.

Example 1.7 (Chain in S^1)**.** Consider the Δ-complex we saw earlier



We have that $C_0^\Delta(S^1) = \mathbb{Z}v$ and that $C_1^\Delta(S^1) = \mathbb{Z}e$. For any $n \geq 2$ we have $C_n^\Delta(S^1) = \{0\}$.

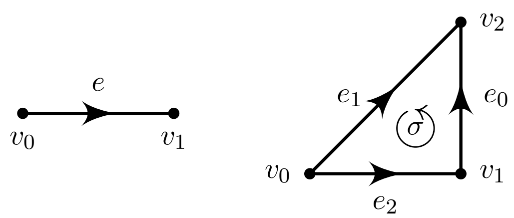
Definition 1.10 (Boundary map)**.** We define the map $\partial_n\colon C_n^\Delta(X) \to C_{n-1}^\Delta(X)$ on basis elements by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

and extend it linearly.

Remark 1.12. The boundary map is a group homomorphism.

Example 1.8. Consider the following as chains in some arbitrary Δ-complex.



Then the boundary of the 1-chain (e) on the left is

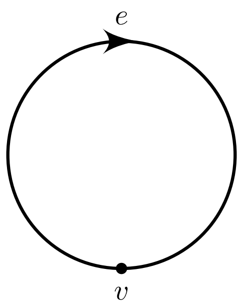
$$\partial(e) = \sigma|_{[v_1]} - \sigma|_{[v_0]} = v_1 - v_0,$$

and the boundary of the 2-chain (σ) on the right is

$$\sigma(\sigma) = \sigma|_{[v_0,v_1]} + \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} = e_2 + e_0 - e_1.$$

Notice that when we calculate the boundary of a chain we don't have do it by the formula, we just have to put a $-$ when the orientation of σ is opposite to the orientation of the edge for example.

Example 1.9. Consider again the Δ-complex of S^1.

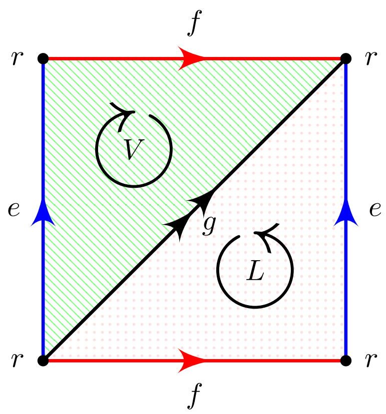


We already computed the chains in S^1 so we know that $\partial\colon \mathbb{Z}e \to \mathbb{Z}v$. However, we see that when $a \in Z$ that

$$\sigma_1(ae) = a\sigma(e) = a(v - v) = 0,$$

which implies that $\sigma_1 = 0$.

Example 1.10. Consider again the Δ-complex of the torus.



The structure of the boundary maps is

$$\{0\} \xrightarrow{\partial_3} \mathbb{Z}V \oplus \mathbb{Z}L \xrightarrow{\partial_2} \mathbb{Z}e \oplus \mathbb{Z}f \oplus \mathbb{Z}g \xrightarrow{\partial_1} \mathbb{Z}r$$

We see that

$$\partial_1(e) = r - r = 0$$
$$\partial_1(f) = r - r = 0$$
$$\partial_1(g) = r - r = 0$$

and

$$\partial_2(V) = e + f - g$$
$$\partial_2(L) = f + e - g$$

Remark 1.13. In the previous example when we computed the boundary of V, we did this by remembering that in a Δ-complex a face is the image of a 2-simplex $[v_0, v_1, v_2]$, which in the case of this complex are all sent to the same point, but from Remark 1.7 we know that the restriction of V to a face is already determined by the orientation of the edges, so that's how we know that $V|_{[v_0,v_1]} = -g$ etc.

Lemma 1.1. *We have that $\partial_{n-1} \circ \partial_n = 0$.*

Proof. First we check for basis elements. Let $\sigma \in C_{n+1}(X)$ be so that $\sigma\colon [v_0, \dots, v_n] \to X$. By definition we have that

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

which implies that $(\partial_{n-1} \circ \partial_n)(\sigma)$ is equal to

$$\sum_{i=0}^{n} (-1)^i \partial_{n-1} \left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_j,\dots,v_n]} + \sum_{j>i} (-1)^{j-1} \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_j,\dots,v_n]} \right)$$

but this is just equal to

$$\sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_j,\dots,v_n]} - \sum_{j>i} (-1)^i (-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_j,\dots,v_n]} = 0$$

□

Definition 1.11 (Cycles)**.** We define the set of cycles to be $Z_n^{\Delta}(X) = \ker \partial_n$. Elements in $Z_n^{\Delta}(X)$ are called cycles or n-cycles.

Definition 1.12 (Boundaries)**.** We define the set of boundaries to be $B_n^{\Delta}(X) = \operatorname{im} \partial_n$. Elements in $B_n^{\Delta}(X)$ are called boundaries.

Corollary 1.2. *From Theorem 1.1 we have that $B_{n+1}^{\Delta}(X) \subseteq Z_n^{\Delta}(X)$.*

Definition 1.13 (Homology)**.** We define the nth homology to be $H_n^{\Delta}(X) = Z_n^{\Delta}(X)/B_{n+1}^{\Delta}(X)$. The elements of $H_n^{\Delta}(X)$ are called homology classes.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \longrightarrow 0$$

Definition 1.14 (Homologous cycles)**.** Let $z_1, z_2 \in Z_n(X)$ be cycles. Then we denote $z_1 \sim z_2$ and say that z_1 and z_2 are homologous if $[z_1] = [z_2] \in H_n(X)$. This is equivalent to saying $z_1 - z_2 \in B_n(X)$ etc. These equivalency classes are somtimes called homology classes.

Remark 1.14. In general we get that $H^1(X)$ is the abelianization of $\pi_1(X)$. We will see what this means later.

1.3 Singular homology

Definition 1.15 (Singular n-simplex)**.** Let X be a topological space. A singular n-simplex is a continuous map $\sigma\colon \Delta^n \to X$.

Remark 1.15. In order to define singular homology we need to define chains and boundaries again. We define

$$C_n(X) := \{\text{the free abelian group with basis the set of singular } n\text{-simplices}\}.$$

We call elements in $C_n(X)$ singular n-chains. Take a moment to appreciate how big this groups is. The boundary map ∂_n is defined similarly,

$$\partial_n(\sigma) := \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

and cycles (Z_n), boundaries (B_n) and homology ($H_n = Z_n/B_n$) are defined in a similar way to how we defined them in simplicial homology as well.

Lemma 1.3. *Let $X = \{*\}$. Then*

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

Proof. For any $n \geq 1$ we see that

$$C_n(X) = \mathbb{Z}\sigma^n$$

where σ^n is the constant map $[v_0, \dots, v_n] \to \{*\}$. When $n = 0$ we also have that $\partial_0(\sigma^0) = 0$ and if $n \geq 1$ then

$$\partial_n \sigma^n = \sum_{i=0}^{n} (-1)^i \underbrace{\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}}_{\sigma^{n-1}} = \begin{cases} \sigma^{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

We have that

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C_3(\{*\}) & \longrightarrow & C_2(\{*\}) & \longrightarrow & C_1(\{*\}) & \longrightarrow & C_0(\{*\}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \| & & \| \\ \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

and then we have that

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

and for $n \geq 1$ we have one of the following

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{\mathbb{Z}}{\mathbb{Z}} \cong \{0\} \quad \text{or} \quad H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{\{0\}}{\{0\}} \cong \{0\}$$

which completes the proof. $\square$

Remark 1.16. Let X and Y be homeomorphic spaces. Then $H_n(X) \cong H_n(Y)$.

Lemma 1.4. *Let $X = \bigsqcup X_\alpha$ where X_α are the connected components of X. Then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$*

Proof. The intuition behind this lemma is that since each $\sigma \colon \Delta^n \to X$ must be continuous, it must only exist within a single unique path component of $\bigsqcup X_\alpha$. This intuition is enough justification for us to say that

$$C_n(X) = \bigoplus_\alpha C_n(X_\alpha).$$

For each $\sigma \in C_n(X_\alpha)$ we have that $\partial_n \sigma \in C_{n-1}(X_\alpha)$ which implies that

$$Z_n(X) = \ker \partial_n = \bigoplus_\alpha Z_n(X_\alpha)$$
$$B_n(X) = \operatorname{im} \partial_{n+1} = \bigoplus_\alpha B_n(X_\alpha)$$

which implies that

$$H_n(X_\alpha) = Z_n/B_n = \bigoplus_\alpha H_n(X_\alpha)$$

which completes the proof. $\square$

Lemma 1.5. *Let X be path connected. Then $H_0(X) \cong \mathbb{Z}$.*

Proof. We have that
$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1}$$
where $\ker \partial_0 = C_0(X)$ is all maps $\sigma\colon \Delta^0 \to X$ and $\operatorname{im} \partial_1$ is the difference of any two points in X (because each two points can be connected by a path) and thus clearly
$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \mathbb{Z}$$
which is generated by any map $\sigma\colon \Delta^0 \to X$ (all other maps are equal to it after modding by $\operatorname{im} \partial_1$). □

Corollary 1.6. *Let X be a topological space. Then $H_0(X)$ is the direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Remark 1.17. Before getting into homotopy invariance, let us consider a different, more rigorous proof for Theorem 1.5 that will give us more motivation for reduced homology.

Proof of Theorem 1.5. We have that
$$\longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0$$
so $\partial_0 = 0$ which means that
$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{C_0(X)}{\operatorname{im} \partial_1}.$$
We now define a new map $\epsilon\colon C_0(X) \to \mathbb{Z}$ by
$$\epsilon\left(\sum_i n_i\sigma_i\right) = \sum_{i=1}^n n_i.$$
This map is obviously surjective if X is nonempty. If we show that $\ker \epsilon = \operatorname{im} \partial_1$ then by the first isomorphism theorem we would have
$$H_0(X) = \frac{C_0(X)}{\operatorname{im} \partial_1} = \frac{C_0(X)}{\ker \epsilon} \cong \mathbb{Z}.$$
First we observe that $\operatorname{im} \partial_1 \subset \ker \epsilon$ since for a singular 1-simplex (a path) we have
$$\epsilon\left(\partial_1(\sigma)\right) = \epsilon\left(\sigma_{[v_1]} - \sigma_{[v_0]}\right) = 1 - 1 = 0.$$
It is left to show that $\ker \epsilon \subset \operatorname{im} \partial_1$. Let $\sum_i n_i\sigma_i \in \ker \epsilon$. This means that
$$\epsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i = 0.$$
Since the σ_i are singular 0-simplices they are simply constant maps to points of X. We can define a path $\tau_i\colon [0,1] \to X$ from a basepoint x_0 to $\sigma_i([v_0])$ and set σ_0 to be the singular 0-simplex with image x_0. We can consider τ_i as singular 1-simplices, maps $\tau_i\colon [v_0, v_1] \to X$, and then we have $\partial(\tau_i) = \sigma_i - \sigma_0$. So since $\sum_i n_i = 0$
$$\partial_1\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i.$$
Thus $\sum_i n_i\sigma_i \in \operatorname{im} \partial_1$, which shows that $\ker \epsilon \subset \operatorname{im} \partial_1$ and completes the proof. □

Definition 1.16 (Chain complex)**.** A chain complex is a sequence

$$\cdots \longrightarrow A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \cdots$$

where A_n are abelian groups such that $\partial_n \circ \partial_{n+1} = 0$.

Remark 1.18. We sometimes denote $\partial_n \circ \partial_{n+1} = 0$ as $\partial^2 = 0$.

Remark 1.19. The equation $\partial_n \circ \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$.

Remark 1.20. Earlier in the alternative proof for Theorem 1.5 we saw (although we didn't mention it directly) the augumented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

which is called the reduced chain complex.

Definition 1.17 (Reduced homology groups)**.** We define the reduced homology groups $\widetilde{H}_n(X)$ as the homology groups of the augmented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is the map we saw in Theorem 1.5. In this reduced way of computing homoloy, a point has trivial homology groups in all dimensions. We also want to require X to be nonempty to avoid having a nontrivial homology group in dimension -1.

Remark 1.21. We have that

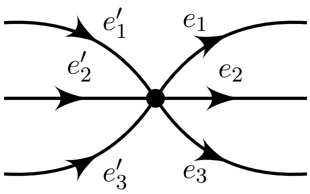
$$H_n(X) = \begin{cases} \widetilde{H}_n(X), & \forall n \geq 1, \\ \widetilde{H}_0(X) \oplus \mathbb{Z}, & n = 0, \end{cases}$$

since $\widetilde{H}_0(X) = 0$ where X is path connected.

Remark 1.22. In order to show that $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$ we notice that ϵ induces a map $\epsilon \colon H_0(X) \to \mathbb{Z}$ whose kernel is in $\widetilde{H}_0(X)$. After some manipulations we get that $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$ as wanted.

Remark 1.23. Formally, we can think of the extra $\mathbb{Z}$ in the augumented chain complex as generated by the unique map $[\emptyset] \to X$ where $[\emptyset]$ is the empty simplex, which no vertices. The augumentation map ϵ is then the usual boundary map since $\partial[v_0] = [\hat{v}_0] = [\emptyset]$.

Remark 1.24 (Geometric interpretation)**.** Let $c = \sum n_\sigma \sigma \in Z_n(X)$ be a cycle. For example we can take $c = \sum n_e e \in Z_1(X)$. Since $0 = \partial c$, every endpoint of a 1-simplex cancels out with an initial point of a 1-simplex. This means that every such point will look somewhat like the following:



Students who major in physics may appreciate that if we define the chain groups to be the span of the singular simplices over $\mathbb{R}$ instead of $\mathbb{Z}$, this looks a lot like Kirchhoff's second law. Moreover, when at every point the amount of edges coming in and going out is equal, the chain is necessarily a cycle, and even a closed 1-manifold. Similarly, we can consider $c \in Z_2(X)$ which is a closed topological surface. That it because, it is a cycle precisely if and only if it has no boundary. When we consider a chain that is not a cycle, we get manifolds with boundary. For reasons beyond my comprehension, this argument does not work in higher dimensions and we only get psuedo-manifolds.

1.4 Homotopy invariance

Definition 1.18. Let $f\colon X \to Y$ be a continuous map. Then for every $\sigma\colon \Delta^n \to X$ the map induces a new map $f \circ \sigma\colon \Delta^n \to Y$. Let $f_\#\colon C_n(X) \to C_n(Y)$ be the map defined on basis elements by $f_\#(\sigma) = f \circ \sigma$. This gives us the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Proposition 1.7. *The map $f_\#$ commutes with ∂. That is, $f_\# \circ \partial = \partial \circ f_\#$. This is equivalent to saying that the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Proof. In the homework. ☐

Definition 1.19 (Chain map)**.** The collection of maps $f_\#$ is called a chain map. Any map similar to $f_\#$ that commutes with the boundary operator is called a chain map.

Proposition 1.8. *A chain map $f_\#$ induces a well defined map homomorphism $f_*\colon H_n(X) \to H_n(Y)$.*

Proof. Let $c \in Z_n(X)$ (this means that $\partial c = 0$). Then since we have

$$\partial f_\#(c) = f_\#(\partial c) = f_\#(0) = 0,$$

we get that $f_\#(c) \in Z_n(Y)$. Similarly for $b \in B_n(X)$ (which means that $b = \partial c$) we get that

$$f_\#(b) = f_\#\left(\partial c\right) = \partial\left(f_\#(c)\right),$$

so since $f_\#(c) \in C_{n+1}(Y)$ we have $f_\#(b) \in B_n(Y)$. Thus $f_\#$ induces a well defined homomorphism $f_*\colon H_n(X) \to H_n(Y)$ which completes the proof. The proof that this is indeed a homomorphism is omitted. ☐

Remark 1.25. The homomorphism f_* is called the induced map of f.

Proposition 1.9 (Functoriality of the induced map)**.** *The induced map satisfies the following two properties for a chain complex*

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

(1) $(fg)_* = f_* g_*$;

(2) $\mathbb{1}_* = \mathbb{1}$ where $\mathbb{1}$ is the identity map of a space or a group.

Corollary 1.10. *Let X and Y be homeomorphic topological spaces. Then $H_n(X) \cong H_n(Y)$.*

Corollary 1.11. *Let $A \subseteq X$ be a retract of X. Then $i_*\colon H_n(A) \to H_n(X)$ is injective and $r_*\colon H_n(X) \to H_n(A)$ is surjective.*

Proof. We have the maps

$$\begin{tikzcd} X \arrow[r, "r"] & A \\ A \arrow[u, "i"] \arrow[ur, "r\circ i"'] & \end{tikzcd}$$

where $r \circ i = \mathrm{id}$ and then from Theorem 1.9 we get

$$\begin{tikzcd} H_n(X) \arrow[r, "r_*"] & H_n(A) \\ H_n(A) \arrow[u, "i_*"] \arrow[ur, "(\mathrm{id})_*"'] & \end{tikzcd}$$

which implies that $r_* \circ i_* = (r \circ i)_* = (\mathrm{id})_* = \mathrm{id}_{H_n(A)}$. This shows that r_* must be surjective and i_* must be injective and completes the proof. ☐

Corollary 1.12. *$\{0,1\} \subseteq [0,1]$ is not a retract.*

Proof. Assume by contradiction that $\{0,1\} \subseteq [0,1]$ is a retract. Then by Theorem 1.11 the map $i_* \colon H_0\left(\{0,1\}\right) \to H_0\left([0,1]\right)$ must be injective. However we know these homology groups so we get that $i_* \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ must be injective. This contradiction completes the proof. ☐

Theorem 1.13. *Let $f, g \colon X \to Y$ be homotopic, that is, $f \simeq g$. Then the induced maps $f_*, g_* \colon H_n(X) \to H_n(Y)$ are equal for all $n \geq 0$.*

Definition 1.20 (Homotopy equivalence)**.** Let X and Y be topological spaces. A continuous map $f \colon X \to Y$ is a homotopy equivalence if there exists a continuous map $g \colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y \quad \text{and} \quad g \circ f \simeq \mathrm{id}_X \,.$$

If there exists a homotopy equivalence between X and Y they are said to be homotopy equivalent (or that they have the same homotopy type), and we denote $X \simeq Y$.

Corollary 1.14. *Let $X \simeq Y$. Then $H_n(X) \cong H_n(Y)$.*

Proof. In the homework. ☐

Example 1.11. We can show that $\mathbb{R}^n \simeq \{*\}$ since we can take the maps

$$\begin{cases} f \colon \mathbb{R}^n \mapsto \{*\}\,, \\ g \colon \{*\} \to 0. \end{cases}$$

Then by Theorem 1.14 we get that

$$H_n\left(\mathbb{R}^n\right) = H_n\left(\{*\}\right) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

Remark 1.26. So far we barely know anything about singular homology. We go back to prove Theorem 1.13 in order to develop our tool set.

Proof of Theorem 1.13. We are going to build a chain homotopy $P \colon C_n(X) \to C_{n+1}(Y)$ such that

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "g_\# - f_\#"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "g_\# - f_\#"'] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d, "g_\# - f_\#"'] \arrow[dl, "P"'] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

and $g_\# - f_\# = \partial P + P\partial$. A map satisfying the above properties is called a *chain homotopy*. Assuming chain homotopy P exists we can show that $f_* = g_*$. First we notice that $P\colon C_n(X) \to C_{n+1}(Y)$. Let $c \in \ker \partial_n$. We then have

$$\underbrace{\partial P(c) + \overbrace{P\partial(c)}^{=0}}_{g_\#(c)-f_\#(c)} = \partial P(c) \in \operatorname{im} \partial_{n+1}$$

which implies that $g_* - f_*$ is the zero map since

$$f_*(c) - g_*(c) = 0 \in H_n(Y) = Z_n(Y)/B_n(Y).$$

Note that in general whenever $g_\# - f_\# = \partial P + P\partial$ we get that $g_* = f_*$. The map we are going to construct P is called the prism map. Our goal is $\partial P + P\partial = g_\# - f_\#$. Since f and g are homotopic there exists a homotopy $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. In general when we construct the map $f_\#$ we have

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \Delta^n \arrow[u, "\sigma"] \arrow[ur, "f\circ\sigma"'] & \end{tikzcd}$$

and we can see how the homotopy naturally induces a similar diagram

$$\begin{tikzcd} X \times [0,1] \arrow[r, "H"] & Y \\ \Delta^n \times [0,1] \arrow[u, "\sigma\times[0,1]"] \arrow[ur, "H\circ(\sigma\times[0,1])"'] & \end{tikzcd}$$

The object $\Delta^n \times [0,1]$ turns out to be a union of $n+1$ copies of Δ^{n+1}. If we denote $[v_0, v_1, \dots, v_n]$ as the vertices of $\Delta^n \times \{0\}$ and $[w_0, w_1, \dots, w_n]$ as the vertices of $\Delta^n \times \{1\}$, these simplices can be written as $\Delta_i^{n+1} = [v_0, \dots, v_i, w_i, \dots, w_n]$ for $0 \le i \le n$. This gives rise to the following definition of $P\colon C_n(X) \to C_{n+1}(Y)$. For $\sigma \in C_n(X)$ let

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_n]}.$$

We now compute $\partial P(\sigma)$.

$$\begin{aligned} \partial P(\sigma) &= \sum_{i=0}^{n} (-1)^i \partial \left(H \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_n]} \right) \\ &= \sum_{j\le i}^{n} (-1)^i (-1)^j H \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} \\ &+ \sum_{j\ge i}^{n} (-1)^i (-1)^{j+1} H \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}. \end{aligned}$$

When $i = j$ the summands almost cancel out as we see below.

$$(-1)^0(-1)^0 \underbrace{H \circ (\sigma \times \mathrm{id})\,|_{[w_0,\dots,w_n]}}_{g_\#(\sigma)} + (-1)^n(-1)^{n+1} \underbrace{H \circ (\sigma \times \mathrm{id})\,|_{[v_0,\dots,v_n]}}_{f_\#(\sigma)} = g_\#(\sigma) - f_\#(\sigma).$$

If we denote

$$(*) = \sum_{j<i}^{n}(-1)^i(-1)^j H\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} + \sum_{j>i}^{n}(-1)^i(-1)^{j+1} H\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

This leaves us with

$$\partial P(\sigma) = (*) + g_\#(\sigma) - f_\#(\sigma).$$

In order to compute $P(\partial\sigma)$ we first notice that

$$\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

We can now compute $P(\partial\sigma)$ as follows:

$$\begin{aligned}
P(\partial\sigma) &= \sum_{i=0}^{n}(-1)^i P\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right)\\
&= (-1)\Bigg(\sum_{j<i}^{n}(-1)^i(-1)^j H\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}\\
&\qquad + \sum_{j>i}^{n}(-1)^i(-1)^{j+1} H\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}\Bigg)\\
&= (-1)(*).
\end{aligned}$$

It follows that

$$\partial P + P\partial = (*) + g_\# - f_\# - (*) = g_\# - f_\#.$$

This completes the proof. □

Corollary 1.15. *Let X be contractible. Then by Theorem 1.14 it follows that $H_n(X) = H_n\big(\{*\}\big)$. Moreover,*

$$H_n\big(\{*\}\big) = \begin{cases} \mathbb{Z}, & n = 0,\\ 0, & n \neq 0.\end{cases}$$

1.5 Exact sequences

Definition 1.21 (Exact sequence)**.** A sequence of abelian groups (or any sequence of objects)

$$\cdots \xrightarrow{\alpha_{i+2}} A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

is said to be exact if $\ker\alpha_i = \operatorname{im}\alpha_{i+1}$ for all i. If $\ker\alpha_i = \operatorname{im}\alpha_{i+1}$ for some i. then it is said to be exact at A_i.

Remark 1.27. A sequence is exact if and only if it is a chain complex with trivial homologies.

Remark 1.28. We call these sequences exact, because while usually we have that $\operatorname{im}\alpha_{n+1} \subseteq \ker\alpha_n$, here the image of α_{n+1} is *exactly* the kernel of α_n. This means that if $a \in A_n$ maps to zero then necessarily there exists $b \in A_{n+1}$ so that $\alpha_{n+1}(b) = a$. We also still have the property $\alpha_{n+1}\circ\alpha_n = 0$.

Example 1.12. The sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B$$

is exact if and only if $\ker\alpha = 0$ if and only if α is injective.

Example 1.13. The sequence

$$A \xrightarrow{\beta} B \longrightarrow 0$$

is exact if and only if $\operatorname{im}\beta = B$ if and only if β is surjective.

Example 1.14. We can combine the last two results and get that the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact if and only if α is an isomorphism.

Example 1.15 (Short exact sequence)**.** A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

which means that α must be injective and β must be surjective. Another way to think about it is that $A \subseteq B$ and that C is a quotient of B. In this way since $\ker\beta = \operatorname{im}\alpha$ we can think about it as $C = B/A$.

Definition 1.22 (Good pair)**.** Let X be a topological space and let $A \subseteq X$. The pair (X, A) is said to be a good pair if A has an open neighbourhood $A \subseteq U \subseteq X$ such that A is a deformation retract of U.

Theorem 1.16. *Let (X, A) be a good pair. Then there exists an exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "q_*"] & \widetilde{H}_n(X/A) \arrow[dlll, rounded corners, "\partial"] & \\
& \widetilde{H}_{n-1}(A) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(X) \arrow[r, "q_*"] & \widetilde{H}_{n-1}(X/A) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \xrightarrow{\partial} \widetilde{H}_0(A) \xrightarrow{i_*} \widetilde{H}_0(X) \xrightarrow{q_*} \widetilde{H}_0(X/A) \longrightarrow 0$$

where $i\colon A \hookrightarrow X$ is the inclusion map and $q\colon X \to X/A$ is the quotient map.

Corollary 1.17. *We have that*

$$\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n, \\ \{0\}, & \textit{otherwise.} \end{cases}$$

Proof. Denote $X = \mathbb{D}^n$ and $A = \partial\mathbb{D}^n$. After verifying that this is a good pair (not too hard) the result of Theorem 1.16 is that there exists an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \widetilde{H}_k(\partial\mathbb{D}^n) \arrow[r, "i_*"] & \widetilde{H}_k(\mathbb{D}^n) \arrow[r, "q_*"] & \widetilde{H}_k(\mathbb{D}^n/\partial\mathbb{D}^n) \arrow[dlll, rounded corners, "\partial"] & \\
& \widetilde{H}_{k-1}(\partial\mathbb{D}^n) \arrow[r, "i_*"] & \widetilde{H}_{k-1}(\mathbb{D}^n) \arrow[r, "q_*"] & \widetilde{H}_{k-1}(\mathbb{D}^n/\partial\mathbb{D}^n) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \xrightarrow{\partial} \widetilde{H}_0(\partial\mathbb{D}^n) \xrightarrow{i_*} \widetilde{H}_0(\mathbb{D}^n) \xrightarrow{q_*} \widetilde{H}_0(\mathbb{D}^n/\partial\mathbb{D}^n) \longrightarrow 0$$

We know that the homology groups of $\mathbb{D}^n$ are trivial so we can consider the exact subsequence

$$0 \longrightarrow \widetilde{H}_k(\mathbb{D}^n/\partial\mathbb{D}^n) \xrightarrow{\partial} \widetilde{H}_{k-1}(\partial\mathbb{D}^n) \longrightarrow 0$$

which is equivalent to

$$0 \longrightarrow \widetilde{H}_k(S^n) \xrightarrow{\partial} \widetilde{H}_{k-1}(S^{n-1}) \longrightarrow 0$$

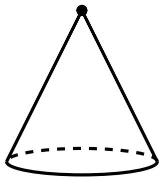
and since this sequence is exact, as we saw in Example 1.14, we have $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$. Then since we know that

$$H_k(S^0) = \begin{cases} \mathbb{Z}, & k = 0, \\ \{0\}, & \text{otherwise}, \end{cases}$$

the result follows by induction. □

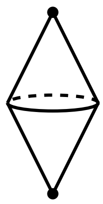
Definition 1.23 (Cone of a space)**.** Let X be a topological space. We define the cone of X to be $CX = X \times [0,1]/X \times \{1\}$.

Example 1.16. Let $X = S^1$. Then the cone of a space is

$CX = CS^1 =$ 

Definition 1.24 (Suspension of a space)**.** Let X be a topological space. We define the suspension of X to be $SX = CX/X = CX/X \times \{1\}$.

Example 1.17. Let $X = S^1$. Then the cone of a space is

$SX = SS^1 =$ 

Remark 1.29. Notice that SS^n is homeomorphic to S^{n+1}. We can now proceed to generalize the statement $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ from Theorem 1.17.

Proposition 1.18. *Let X be a topological space. Then $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.*

Proof. We know that $CX/X = SX$. After verifying that (CX, X) is a good pair (not too hard) the result of Theorem 1.16 is that there exists an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \widetilde{H}_n(X) \arrow[r, "i_*"] & \widetilde{H}_n(CX) \arrow[r, "q_*"] & \widetilde{H}_n(SX) \arrow[dlll, "\partial"] \\
\widetilde{H}_{n-1}(X) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(CX) \arrow[r, "q_*"] & \widetilde{H}_{n-1}(SX) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \xrightarrow{\partial} \widetilde{H}_0(X) \xrightarrow{i_*} \widetilde{H}_0(CX) \xrightarrow{q_*} \widetilde{H}_0(XS) \longrightarrow 0$$

We know that the homology groups of CX are trivial because CX is contracible to the point $X \times \{0\}$. This allows us to consider the exact subsequence

$$0 \longrightarrow \widetilde{H}_n(SX) \xrightarrow{\partial} \widetilde{H}_{n-1}(X) \longrightarrow 0$$

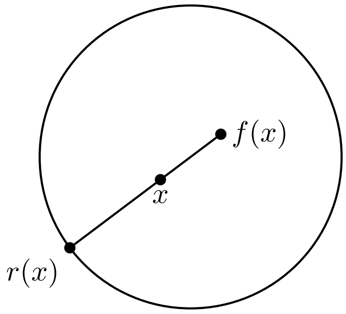
and since this sequence is exact, as we saw in Example 1.14, we have $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ which completes the proof. □

Corollary 1.19. *$\partial\mathbb{D}^n$ is not a retract of $\mathbb{D}^n$.*

Proof. Suppose that $r\colon \mathbb{D}^n \to \partial\mathbb{D}^n$ is a retraction. Then $r_*\colon H_{n-1}(\mathbb{D}^n) \to H_{n-1}(\partial\mathbb{D}^n)$ is surjective, but using previous results $r_*\colon 0 \to \mathbb{Z}$ which is a contradiction and completes the proof. □

Corollary 1.20 (Brouwer's fixed point theorem)**.** *Every contractive map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.*

Proof. Assume, by way of contradiction, that $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has to fixed points. Define $r\colon \mathbb{D}^n \to \partial\mathbb{D}^n$ by mapping x to the intersection of the ray from $f(x)$ through x to $\partial\mathbb{D}^n$.



The map $r\colon \mathbb{D}^n \to \partial\mathbb{D}^n$ is continuous and $r|_{\partial\mathbb{D}^n} = \mathrm{id}$ so it is a retraction. Then by Theorem 1.11 the induced map r_* must be surjective for all homology groups. In particular $r_*\colon H_{n-1}(\mathbb{D}^n) \to H_{n-1}(\partial\mathbb{D}^n) = H_{n-1}(S^{n-1})$ must be surjective. However, we have already computed these homology groups, so we know that r_* is a map from $\{0\}$ to $\mathbb{Z}$ so it can't be surjective. This contradiction completes the proof. □

1.6 Relative homology

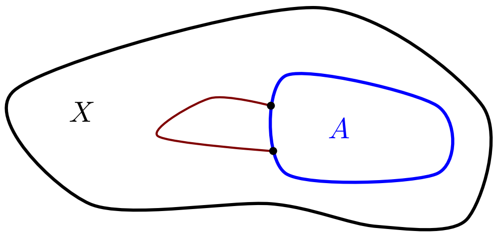
Definition 1.25 (Relative homology)**.** Let (X, A) be a pair such that $A \subseteq X$ where X is a topological space. We have already seen that $C_n(A) \subseteq C_n(X)$. We define $C_n(X, A) := C_n(X)/C_n(A)$. This is the free abelian group with basis $\{\sigma_\alpha\colon \Delta^n \to X\}_\alpha$ such that $\sigma_\alpha(\Delta^n) \nsubseteq A$.

Remark 1.30. Note that $\partial\colon C_n(X) \to C_{n-1}(X)$ takes $\partial\left(C_n(A)\right) \subseteq C_{n-1}(A)$, and ∂ induces a map $\partial\colon C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$. This means that the boundary map $\partial\colon C_n(X, A) \to\colon C_{n-1}(X, A)$ is well defined.

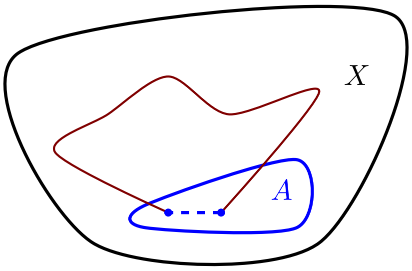
Remark 1.31. The pair $\left(C_n(X, A), \partial\right)$ is a chain complex because $\partial^2 = 0$.

Definition 1.26 (Relative homology)**.** We define the homologies of (X, A) (relative homology groups) to be the homologies of the chain complex $\left(C_n(X, A), \partial\right)$. We denote the relative homology groups with $H_n(X, A)$. This means that $H_n(X, A) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ for the boundary of the chain complex $\left(C_n(X, A), \partial\right)$.

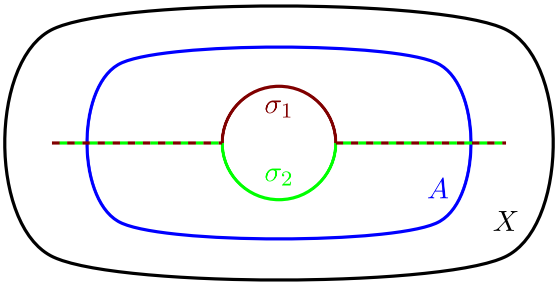
Remark 1.32. A relative cycle is a relative chain $c + C_n(A) \in C_n(X, A)$ for some $c \in C_n(X)$ such that $\partial(c) \in C_{n-1}(A)$. Below in red is an example of a relative 1-cycle.



Remark 1.33. A relative boundary is a relative chain $b + C_n(A) \in C_n(X, A)$ for some $b \in C_n(X)$ such that $b + C_n(A) = \partial\left(c + C_{n+1}(A)\right) = \partial c + C_n(A)$ for some $c + C_{n+1}(A) \in C_{n+1}(X, A)$. This is equialent to $b - \partial c \in C_n(A)$. Below in red is an example of a relative 2-boundary.



Remark 1.34. The following red and green 1-chains



are not equal in $C_1(X, A)$. Recall that the sums of the singular chain groups are formal, this means that this is not a matter of pointwise function addition or anything, and certainly $\sigma_1 - \sigma_2$ is not an element of $C_n(A)$.

Definition 1.27 (Map of pairs)**.** A map $f\colon (X, A) \to (Y, B)$ is said to be a map of pairs if it is continuous and $f(A) \subseteq B$.

Remark 1.35. A map of pairs f induces a map on the relative homology groups. We denote it simply as $f_*\colon H_n(X, A) \to H_n(Y, B)$. Moreover, if $f, g\colon (X, A) \to (Y, B)$ and $f \simeq_A g$ then $f_* = g_*$. In order to show this we just notice that the prism operator P from Theorem 1.13 takes $C_n(A)$ to $C_{n+1}(B)$, so it induces a relative prism operator $P\colon C_n(X, A) \to C_{n+1}(Y, B)$. Since the relative pris, operator is simply the prism operator reduced to quotient groups the formula $\partial P + P\partial = g_\# + f_\#$ is still satisfied, hence the maps $f_\#$ and $g_\#$ are homotopic on the relative chain groups, so they induce the same homomorphism on the relative homology groups.

Remark 1.36. The rows of the following diagram are short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_\#"] \arrow[d, "\partial"'] & C_n(X) \arrow[r, "q_\#"] \arrow[d, "\partial"'] & C_n(X, A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_\#"] & C_{n-1}(X) \arrow[r, "q_\#"] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

and moreover, the diagram is commutative. This means that both $i_\#$ and $q_\#$ are chain maps. We can also keep adding rows to this diagram by considering more levels of chain groups. In this way, we get a short exact sequences of chain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{q} \mathcal{C} \longrightarrow 0$$

where $\mathcal{A}$ denotes the chain complexes of A, $\mathcal{B}$ denotes the chain complexes of X and $\mathcal{C}$ denotes the chain complexes of (X, A).

Theorem 1.21. *Let*

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there exists a long exact sequence (l.e.s) in homology

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{j_*} H_n(\mathcal{C}) \xrightarrow{\partial} H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

Remark 1.37. With regards to our previous example i is the inclusion map and j is the quotient map.

Corollary 1.22. *If (X, A) is a good pair, then the sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

is exact.

Proof of Theorem 1.21. The short exact sequence of chain complexes can be written as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since i and j are chain maps, they commute with the boundary maps and thus this diagram commutes. In order to obtain the sequence

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{j_*} H_n(\mathcal{C}) \xrightarrow{\partial} H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

we need to have a map between $H_n(\mathcal{C})$ and $H_{n-1}(\mathcal{A})$. That's a map $\partial\colon C_n \to A_{n-1}$. Let $c \in C_n$. Since j is surjective there exists $b \in B_n$ so that $j(b) = c$. We can then apply the boundary map on b and send it to B_{n-1}. We now want to show that there exists $a \in A_{n-1}$ so that $i(a) = \partial(b)$, since this will allow us to map c to that a. This means we need to show that $\partial(b) \in \operatorname{im}(i)$. By exactness we have $\operatorname{im}(i) = \ker(j)$ so we need to show that $j(\partial b) = 0$. Since j commutes with ∂ we know that $j(\partial b) = \partial j(b) = \partial c$. This means that $\partial(b) \in \operatorname{im}(i)$ if and only if $c \in \ker \partial_n$. This suffices to show the existence of a map between $H_n(\mathcal{C})$ and $H_{n-1}(\mathcal{A})$. That is because $H_n(\mathcal{C}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ so we only care about where we map elements of the kernel. We now have a map that maps $c \in H_n(\mathcal{C})$ to some $a \in H_{n-1}(\mathcal{A})$. We need to make sure that $a \in \ker \partial_{n-1}$ in order for the map to be well defined. Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-2} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "\partial"] & B_{n-1} \arrow[r, "\partial"] & B_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Since $a \in A_{n-1}$ we have $i(\partial a) \in B_{n-2}$. Since i is a chain map $i(\partial a) = \partial i(a)$. We know from earlier that $i(a) = \partial b$ so we get that $i(\partial a) = \partial\partial b = 0$. This implies that $\partial a \in \ker(i)$, but since

i is injective $\partial a = 0$. More precisely $\partial_{n-1} a = 0$ so $a \in \ker \partial_{n-1}$ so the map from $H_n(\mathcal{C})$ to $H_{n-1}(\mathcal{A})$ is well defined. It is left to show that the sequence

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{j_*} H_n(\mathcal{C}) \xrightarrow{\partial} H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

is exact. The arguemnts so far involved a lot of what's called *diagram chasing*. I am sure I don't have to explain why that is. In order to complete the proof, we just show that the long sequence is exact, and we do this by definition. This means we do this by diagram chasing. Moreover, that diagram chasing is omitted from this proof. $\square$

Remark 1.38. There is a version of Theorem 1.21 for reduced homology.

Example 1.18. Let $x \in X$. Then from Theorem 1.21 applied on $(X, \{x\})$ we get that there exists a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n\left(\{x\}\right) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{q_*} \widetilde{H}_n(X, x_0) \xrightarrow{\partial} \widetilde{H}_{n-1}\left(\{x\}\right) \longrightarrow \cdots$$

and since we know that $\widetilde{H}_n\left(\{x\}\right) = \widetilde{H}_{n-1}\left(\{x\}\right) \cong \{0\}$ and since this sequence is exact we get that $\widetilde{H}_n(X) \cong \widetilde{H}_n(X, x_0)$.

Remark 1.39 (Long exact sequence of a triple)**.** We can generalize the long exact sequence of a pair (X, A) the a long exact sequence of a triple (X, A, B), where $B \subset A \subset X$:

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A, B) \longrightarrow \cdots$$

This is just the long exact sequence of homology groups associated with the short exact sequence of chain complexes formed by the short exact sequences

$$0 \longrightarrow C_n(A, B) \longrightarrow C_n(X, B) \longrightarrow C_n(X, A) \longrightarrow 0$$

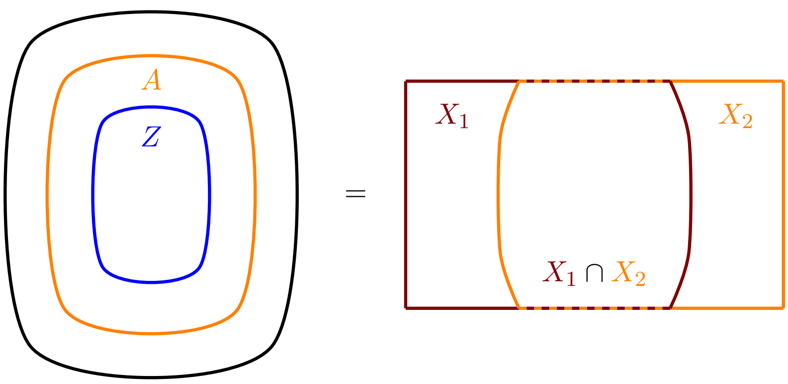
For example, if b is a point, the long exact sequence of the triple (X, A, B) becomes the long exact sequence of reduced homology for the pair (X, A).

1.7 Excisions

Theorem 1.23 (Excision)**.** *Let $Z \subseteq A \subseteq X$ so that $\overline{Z} \subseteq \mathring{A}$. Then the inclusion map $i\colon (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X, A) \cong H_n(X - Z, A - Z)$ for all n.*

Corollary 1.24. *Let (X, A) be a good pair. Then $H_n(X, A) \simeq \widetilde{H}_n(X/A)$.*

Remark 1.40. An equivalent phrasing of the excision theorem is that if $X_1, X_2 \subseteq X$ and $\mathring{X}_1 \cup \mathring{X}_2 = X$, then the inclusion $i\colon (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism $H_n(X_1, X_1 \cap X_2) \cong H_n(X, X_2)$ for all n. In this case $A = X_2$ and $Z = X - X_1 = X_2 - X_1$ etc. Below are diagrams showing the equivalency of the theorems



$$A = X_2$$
$$Z = X - X_1 = X_2 - X_1$$
$$X - Z = X_1$$
$$A - Z = X_1 \cap X_2$$

Definition 1.28 ($\mathcal{U}$-chains)**.** Let $\mathcal{U} = \{U_j\}$ be a collection of subsets of X such that their interiors form an open cover for X, that is, $X = \bigcup_{U \in \mathcal{U}} \mathring{U}$. Then we write

$$C_n^{\mathcal{U}}(X) = \operatorname{span}\left\{\sigma \colon \Delta^n \to X \mid \exists U \in \mathcal{U} \text{ s.t. } \sigma(\Delta^n) \subseteq U\right\}.$$

Remark 1.41. Since $\partial(C_n^U(X)) \subseteq C_{n-1}^U(X)$, the groups $C_n^{\mathcal{U}}(X)$ form a chain complex. We denote the homology of this chain complex by $H_n^{\mathcal{U}}(X)$.

Proposition 1.25. *The inclusion map $\iota \colon C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a homotopy equivalence between $C_n^{\mathcal{U}}(X)$ and $C_n(X)$. That is, there exists $\rho \colon C_n(X) \to C_n^{\mathcal{U}}(X)$, such that $\iota \circ \rho$ and $\rho \circ \iota$ are chain homotopic to the identity. Hence ι induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.*

Proof. We will prove this proposition in four steps.

(1) *Barycentric Subdivision of Simplicies.* Recall that any n-simplex given by $[v_0, v_1, \ldots, v_n]$ is defined to be all the points $\sum t_i v_i$ so that $t_i \geq 0$ and $\sum t_i = 1$. We define the barycenter[1] of an n-simplex to be the point with equal barycentric coordinates $b = \sum_{i=0}^n \frac{1}{n+1} v_i$. We define the barycenter subdivision of an n-simplex inductively. For $n = 0$ we define the subdivision as just the point itself

$$BS\left([v_0]\right) = \left\{[v_0]\right\}.$$

We can now define the subdivision of a n-simplex for $n = 1$ as follows:

$$BS\left([v_0, v_1]\right) = \bigcup_{i=0}^{1} \bigcup_{s \in BS\,(\Delta^i)} \left\{[b, s]\right\} = \left\{[b, v_0], [b, v_1]\right\},$$

where Δ^i represents the ith face of the Δ-complex, and b is constant, it is the barycenter of the 1-simplex. Inductively, we can assume that barycentric subdivision is defined for an $(n-1)$-simplex, and define

$$BS\left([v_0, v_1, \ldots, v_n]\right) = \bigcup_{i=0}^{n} \bigcup_{s \in BS\,(\Delta^i)} \left\{[b, s]\right\}.$$

Note that the barycentric division of an n-simplex is $(n+1)!$ n-simplices who cover the original Δ-simplex. In particular $|BS([v_0, v_1, \ldots, v_n])| = (n+1)!$. Although this is not important for the proof, $BS(\Delta^n)$ does form a simplicial complex structure on Δ^n. We want to show that

$$\operatorname{diam}[w_0, \ldots, w_n] \leq \frac{n}{n+1} \operatorname{diam}[v_0, \ldots, v_n].$$

for each $[w_0, \ldots, w_n] \in BS(\Delta^n)$. We first observe that

$$\operatorname{diam}[w_0, \ldots, w_n] = \max\left\|w_i - w_j\right\|$$

becauase the distance of any two points v and $\sum_i t_i v_i$ always satisfies

$$\left| v - \sum_{i=0}^n t_i v_i \right| = \left| \sum_{i=0}^n t_i (v - v_i) \right| \leq \sum_{i=0}^n t_i |v - v_i| \leq \sum_{i=0}^n t_i \max_j |v - v_j| = \max_j |v - v_j|.$$

The rest of the proof is geometric. Its significance is that it allows us to divide any n-simplex into smaller simplicies with arbirarily small diameter.

[1] bary means mass

(2) The rest of the proof might be added eventually.

□

Theorem 1.26 (Excision)**.** *Let X be a topological space so that $A, B \subseteq X$ and $\mathring{A} \cup \mathring{B} = X$. Then the inclusion map $i\colon (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_n(B, A \cap B) \cong H_n(X, A)$.*

Proof. dependent on proof of excision. to be added. □

Theorem 1.27. *Let (X, A) be a good pair. Then the quotient map $q\colon (X, A) \to (X/A, A/A)$ induces an isomorphism $q_*\colon H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$ for all n.*

Proof. We have seen that a space relative to a point is isomorphic to the reduced homology of the space. Therefore we only need to show an isomorphism $q_*\colon H_n(X, A) \to H_n(X/A, A/A)$. Let U be a neighbourhood of A in X that deformation retracts onto A. We have the commutative diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "q_*"] & H_n(X-A, U-A) \arrow[l] \arrow[d, "q_*"] \\ H_n(X/A, A/A) \arrow[r] & H_n(X/A, U/A) & H_n(X/A - A/A, U/A - A/A) \arrow[l] \end{tikzcd}$$

In the long exact sequence of the triple (X, U, A) we get that

$$\cdots \longrightarrow H_n(U,A) \longrightarrow H_n(X,A) \longrightarrow H_n(X,U) \longrightarrow H_{n-1}(U,A) \longrightarrow \cdots$$

and since a deformation retraction of U onto A gives a homotopy equivalence of pairs $(U, A) \simeq (A, A)$ we have that $H_n(U, A) \cong H_n(A, A)$ for all n, so $H_n(U, A) \cong H_n(U, A) \cong \{0\}$ and thus from the exactness of the sequence $H_n(X, A) \cong H_n(X, U)$ so we may assume the upper left horizontal arrow in our commutative diagram is an isomorphism. The deformation retraction from U to A induces a deformation retraction of U/A onto A/A, so the same argument shows that we may assume the lower left horizontal arrow in our commutative diagram is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right vertical map q_* is an isomorphism since we can see by recalling point set topology that q restricts to a homeomorphism on the complement of A. From the commutativity of the diagram it follows that the middle q_* and then also the left q_* is an isomorphism. □

Theorem 1.28. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be homeomorphic open sets. Then $m = n$.*

Proof. We want to compute the local homology at $x \in U$

$$H_k(X, X - \{x\}).$$

From the excision theorem we have that

$$\widetilde{H}_k(U, U - \{x\}) \cong \widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}).$$

We consider the long exact sequence of $\mathbb{R}^n - \{x\} = A \subset X = \mathbb{R}^n$,

$$\widetilde{H}_k(\mathbb{R}^n) \longrightarrow \widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^n - \{x\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^n) \ .$$

We know that $\widetilde{H}_k(\mathbb{R}^n) = \widetilde{H}_{k-1}(\mathbb{R}^n) = \{0\}$ which implies that $\widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n - \{x\})$. Moreover, we can now combine the result that $\widetilde{H}_{k-1}(\mathbb{R}^n - \{x\}) \cong \widetilde{H}_{k-1}(S^{n-1})$ and the result from the excision theorem to get

$$\widetilde{H}_k(U, U - \{x\}) \cong \widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \widetilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & k = n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $\varphi\colon U \to V$ be a homeomorphism. By φ_* we get $\widetilde{H}_k(U, U - \{x\}) \cong H_k(V, V - \{\varphi(x)\})$. We can repeat the same process from earlier and get that

$$\widetilde{H}_k(V, V - \{\varphi(x)\}) \cong \widetilde{H}_{k-1}(S^{m-1}) = \begin{cases} \mathbb{Z}, & k = m, \\ \{0\}, & \text{otherwise,} \end{cases}$$

which implies that $n = m$. □

Remark 1.42. The result of Theorem 1.28 is also sometimes referred to as "invariance of dimension".

Remark 1.43. In the proof of Theorem 1.28 we have seen the concept of *local homology*. That is, the homology group $H_n(X, X - \{x\})$. The reason this is called local homology, is because as long as we assume $\{x\}$ is closed, we get from excision that $H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$. This shows that the structure of this group depends only on the local topology of X around x. Since a homeomorphism must induce isomorphisms $H_n(X, X - \{x\}) \cong H_n(Y, Y - \{f(x)\})$ for all x and all n, the local homology groups can be used to tell when spaces are not locally homeomorphic at certain points, as in Theorem 1.28.

Remark 1.44. We have seen that

$$H_k(\mathbb{D}^n, \partial\mathbb{D}^n) \cong \widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & n = k, \\ \{0\}, & \text{otherwise.} \end{cases}$$

In the following proposition we will find the explicit cycle that generates $H_n(S^n) \cong H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$ where we identify $H_n(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_n(\Delta^n, \partial\Delta^n)$.

Proposition 1.29. *The map $\sigma_n\colon \Delta^n \to \Delta^n$ where $\sigma_n = \mathrm{id}_n$ viewed as a singular n-simplex is the generator of $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.*

Proof. Note that since $\partial\sigma_n \in C_{n-1}(\partial\Delta^n)$ we have $\partial\sigma_n = 0 \in C_{n-1}(\Delta^n)/C_{n-1}(\partial\Delta^n)$ and thus σ_n is a cycle so $[\sigma_n] \in H_n(\Delta^n, \partial\Delta^n)$. We will prove the proposition by induction on n.

When $n = 0$ then σ is clearly the generator of $H_0(\Delta^0/\partial\Delta^0) = H_0(\{*\}/\emptyset) = H_0(\{*\})$.

When $n > 0$ let $\Lambda = \partial\Delta^n - \mathring{\Delta}_i$ where Δ_i is some face of Δ^n (please appreciate the incredible notation). We claim that there exist isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\cong} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

Consider the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$

$$\cdots \longrightarrow H_n(\partial\Delta^n, \Lambda) \longrightarrow H_n(\Delta^n, \Lambda) \longrightarrow H_n(\Delta^n, \partial\Delta^n) \longrightarrow H_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow \cdots$$

We know that elements of the form $H_i(\Delta^n, \Lambda)$ are the zero group since Δ^n deformation retracts onto Λ, so $(\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$ and $H_i(\Delta^n, \Lambda)$ is trivial. By exactness of the long exact sequence we get the first isomorphism $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$.

The second isomorphism is induced by the inclusion $i\colon \Delta^{n-1} \hookrightarrow \partial\Delta^n$ as the face not contained in Λ. When $n = 1$, the inclusion clearly induces an isomorphism between $H_0(\Delta^0, \partial\Delta^0)$ and $H_0(\partial\Delta^n, \Lambda)$. When $n > 1$, the set $\partial\Delta^{n-1}$ is nonempty so both $(\partial\Delta^n, \Lambda)$ and $(\Delta^{n-1}, \partial\Delta^{n-1})$ are good pairs. Then i induces a homeomorphism of the quotients $\partial\Delta^n/\Lambda \cong \Delta^{n-1}/\partial\Delta^{n-1}$. This shows the existence of the second isomorphism.

We get that $[\sigma]$ is sent (under the first isomorphism) to $[\partial\sigma] = \pm[\sigma_{n-1}]$ where σ_{n-1} is the identity on one of the faces of Δ^n.

By the inductive hypothesis σ_{n-1} is a generator of $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$ so $\partial\sigma_n$ is a generator of $H_n(\partial\Delta^n, \Lambda)$ and hence σ_n is a generator of $H_n(\Delta^n, \partial\Delta^n)$ (because they are images of each other through the isomorphisms).

In order to describe the cycle generating $\widetilde{H}_n(S^n)$ (this is very similar to a problem from the homework) we first identify $S^n \cong \Delta_1^n \cup \Delta_2^n / \partial\Delta_1^n \sim \partial\Delta_2^n$ so that the boundaries are glued in the obvious way (the one that preserves the order of the vertices). Let $\sigma_{1,2}$ be the singular n-simplices that represent $\Delta_{1,2}^n$. We claim that $[\sigma_1 - \sigma_2]$ is the generator of $\widetilde{H}_n(S^n)$.

We have that $[\sigma_1 - \sigma_2] \in \widetilde{H}_n(S^n)$. Consider the isomorphisms

$$\widetilde{H}_n(S^n) \xrightarrow{\cong} \widetilde{H}_n(S^n, \Delta_2^n) \xleftarrow{\cong} \widetilde{H}_n(\Delta_1^n, \partial\Delta_1^n)$$

The first isomorphism comes from the long exact sequence of the pair (S^n, Δ_2^n) and the second isomorphism is justified in the nontrivial cases $n > 0$ by passing to quotients as before. Under these isomorphisms the cycle $\sigma_1 - \sigma_2$ in the first group corresponds to the cycle Δ_1^n in the third group, which represents a generator of this group as we have seen, so $\Delta_1^n - \Delta_2^n$ represents a generator of $\widetilde{H}_n(S^n)$. □

Proposition 1.30. *Let $\{(X_\alpha, x_\alpha)\}_\alpha$ be a collection of good pairs. Then the inclusion maps $i_\alpha \colon X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism*

$$\bigoplus_\alpha (i_\alpha)_* \colon \widetilde{H}_k\left(\bigvee_\alpha (X_\alpha, x_\alpha)\right) \xrightarrow{\cong} \bigoplus_\alpha \widetilde{H}_k(X_\alpha).$$

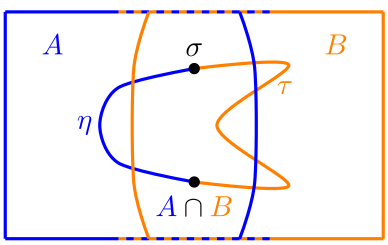
Proof. Notice that the pair $(\bigsqcup X_\alpha, \{x_\alpha \mid \alpha\})$ is a good pair. Since reduced homology is the same as homology relative to a basepoint, the result follows from Theorem 1.27 by taking $(X, A) = (\bigsqcup X_\alpha, \{x_\alpha \mid \alpha\})$. Then

$$\begin{aligned}\widetilde{H}_k\left(\bigsqcup_\alpha X_\alpha / \{x_\alpha \mid \alpha\}\right) &\cong H_k\left(\bigsqcup_\alpha X_\alpha, \{x_\alpha\}_\alpha\right) \\ &= \bigoplus_\alpha H_k(X_\alpha, x_\alpha) \\ &= \bigoplus_\alpha \widetilde{H}_k(X_\alpha).\end{aligned}$$

□

1.8 Mayer–Vietoris

Theorem 1.31. *Let $X = \mathring{A} \cup \mathring{B}$ where $A, B \subseteq X$. Consider the following diagram:*



Then

$$([\tau], [\eta]) \longmapsto [\tau] + [\eta]$$

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

$$[\sigma] \longmapsto ([\sigma], -[\sigma]) \qquad [\sigma] = [\tau] + [\eta] \longmapsto \partial\tau$$

is a long exact sequence.

Proof. Let $\mathcal{U} = \{A, B\}$. The sequence

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{f} C_n(A) \oplus C_n(B) \xrightarrow{g} C_n^{\mathcal{U}}(X) \longrightarrow 0$$

where $f \colon \sigma \mapsto (\sigma, -\sigma)$ and $g \colon (\sigma, \eta) \mapsto \sigma + \eta$ is a short exact sequence of chains. Since $H_n(X) \cong H_n^{\mathcal{U}}(X)$ and this is a short exact sequence, the long exact sequence in homology is the result, which concludes the proof. $\square$

Remark 1.45. We can do the same with exact sequences of reduced homology.

Example 1.19. We want to find $\widetilde{H}_k(S^n)$. Let $S^n = \mathring{A} \cup \mathring{B}$ where $A = S^n - \{N\}$ and $B = S^n - \{S\}$ where $\{S, N\}$ are the south and north poles of the n-sphere.

We know that $A \cong B \cong \mathbb{R}^n$ so they are both contractible. We also see that $A \cap B \simeq_{\mathrm{dr}} S^{n-1}$ since there exists a deformation retraction from $A \cap B$ to the equator, which is S^{n-1}.

By the Mayer–Vietoris theorem we get that

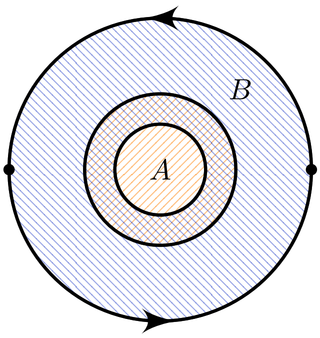
$$\widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \longrightarrow \widetilde{H}_k(S^n) \longrightarrow \widetilde{H}_{k-1}(A \cap B) \longrightarrow \widetilde{H}_{k-1}(A) \oplus \widetilde{H}_{k-1}(B)$$

and by our calculations this is equal to

$$\{0\} \oplus \{0\} \longrightarrow \widetilde{H}_k(S^n) \longrightarrow \widetilde{H}_{k-1}(S^{n-1}) \longrightarrow \{0\} \oplus \{0\}$$

so $H_k(S^n) \cong H_{k-1}(S^{n-1})$. This allows us to compute the homologies of the sphere inductively like we have done in Theorem 1.17.

Example 1.20. Consider the space $\mathbb{RP}^2$. We can choose $A \cong \mathbb{D}^2$ and $B \cong M$ where M is the Möbius band.



Then from Mayer–Vietoris we get the following long exact sequence:

$$\begin{array}{l} \cdots \longrightarrow \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(\mathbb{RP}^2) \longrightarrow \widetilde{H}_1(A \cap B) \longrightarrow \\ \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(\mathbb{RP}^2) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \cdots \end{array}$$

We know that A is contractible so $\widetilde{H}_1(A) = \widetilde{H}_2(A) = \{0\}$ and the Möbius band has a deformation retraction to S^1. So does $A \cap B$. We now have an exact sequence of the form

$$0 \longrightarrow \widetilde{H}_2(\mathbb{RP}^2) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_1(\mathbb{RP}^2) \longrightarrow 0$$

Since we know what the maps in the Mayer–Vietoris sequence are, we can see that the generator of the left $\mathbb{Z}$ is sent to twice the generator of the right $\mathbb{Z}$. This means that $\widetilde{H}_2(\mathbb{RP}^2) = 0$ and that $\widetilde{H}_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

1.9 Naturality

Remark 1.46. The etymology of "naturality" will come after we understand its categorical interpretation.

Definition 1.29 (Naturality)**.** To say that he long exact sequence of a pair is *natural* means that for a map $f\colon (X,A) \to (Y,B)$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "q_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Remark 1.47. There is also a more general algebraic fact that the long exact sequence of homology groups associated with a short exact sequence of chain complexes is natural. This means that that diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\mathcal{A}) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(\mathcal{B}) \arrow[r, "q_*"] \arrow[d, "\beta_*"] & H_n(\mathcal{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(\mathcal{A}) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r] & H_n(\mathcal{A}') \arrow[r, "i'_*"] & H_n(\mathcal{B}') \arrow[r, "q'_*"] & H_n(\mathcal{C}') \arrow[r, "\partial"] & H_{n-1}(\mathcal{A}') \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

1.10 Simplicial vs Singular homology

Remark 1.48. We can define relative simplicial homology in an analogous way to how we defined relative singular homology.

Definition 1.30 (Skeleton of a simplicial complex)**.** Let X be a Δ-complex. Then the k-skeleton of X is given by the union of all images of i-simplices for all $0 \le i \le k$ and is denoted by $X^{(k)}$ or simply X^k.

Proposition 1.32. *The map $f\colon C_n^\Delta(X) \to C_n(X)$ given by the natural map $f\colon \sigma_\alpha \mapsto \sigma_\alpha$ is a chain map and f_* is an isomorphism in homotopy. In particular, the simplicial homology of a Δ-complex for a topological space X is idependent of the choice of the complex.*

Proof. We assume that $X = X^{(n)}$ for some $n \ge 0$ and prove this proposition by induction on the k-skeletons of a given Δ-complex of X. In case $k = 0$ the 0-skeleton X^0 is a disjoint union of points. We already know that

$$H_n^\Delta(X^0) \cong H_n(X^0) \cong \begin{cases} 0, & n > 0, \\ \mathbb{Z}^{\#\text{points}}, & n = 0. \end{cases}$$

Assume that $H_n^\Delta(X^{k-1}) \cong H_n(X^{k-1})$ for all n. We will show that $H_n^\Delta(X^k) \cong H_n(X^k)$ for all n. Notice that the pair (X^k, X^{k-1}) is a good pair. This gives a long exact sequences of simplicial and singular homology, which we can connect by using the induced map.

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "f_*"] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "f_*"] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "f_*"] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "f_*"] & H_{n-1}^\Delta(X^{k-1}) \arrow[d, "f_*"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

By naturality, this diagram commutes. We only regard the vertical arrows from now on, thus the "third arrow" will always refer to the third vertical arrow etc. We know from the induction hypothesis that the second and fifth arrows are isomorphisms. We will now show that the first and fourth arrows are isomorphisms. This can be done by showing that $H_n^\Delta(X^k, X^{k-1}) \cong H_n(X^k, X^{k-1})$ for all n and all k. The simplicial chain group $C_n^\Delta(X^k, X^{k-1})$ is zero for $n \neq k$, and is the free abelian group with basis the k-simplices of X when $n = k$. Hence, $H_n^\Delta(X^k, X^{k-1})$ has exactly the same description:

$$H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} \bigoplus \mathbb{Z}[\sigma_\alpha], & n_\alpha = k \\ 0, & n_\alpha \neq 0. \end{cases}$$

Since X^{k-1} is a strong deformation retract of its neighbourhood in X^k we get that

$$H_n(X^k, X^{k-1}) \cong \widetilde{H}_n(X^k/X^{k-1}) \cong \begin{cases} \operatorname{span}_{\mathbb{Z}} \left\{\sigma_\alpha \colon \Delta^{n_\alpha} \to X \mid n_\alpha = n\right\}, & n = k, \\ 0, & n \neq 0. \end{cases}$$

where the last step follows from the fact that $X^k/X^{k-1} \cong \bigvee_{s_k} S^k$ where s_k is the number of simplices of dimension k. Therefore, we have an isomorphism $H_n^\Delta(X^k, X^{k-1}) \cong H_n(X^k, X^{k-1})$. The five lemma (Theorem 1.33) completes the proof. □

Lemma 1.33 (Five lemma)**.** *Suppse we have five maps α, β, γ, δ and ϵ between two exact sequence*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Then if α, β, δ and ϵ are isomorphisms, then γ is an isomorphism

Proof. diagram chasing. to be added. □

Lemma 1.34. *Let $C \subseteq X$ be compact. Then C meets finitely many simplices of X in their interior. In particular $C \subseteq X^{(k)}$ for some k.*

Proof. Suppose that C interesected infinitely many simplices in their interior. Then there exists an infinite sequence of points x_i each lying in a different open simplex. Then the sets

$$U_i = X - \bigcup_{j \neq i} \{x_j\},$$

form an open cover of C with no finite subcover. □

Remark 1.49. We can use the previous lemma to show Theorem 1.32 without the assumption that $X = X^{(n)}$ for some n. Given an element in $H_n(X)$, it can be represented by a singular n-cycle z, which is the continuous image of finitely many compact simplices, and is thus compact. Then from Theorem 1.34 it intersects with finitely many simplices in X, so it is contained in $X^{(k)}$ for some k, and since $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, we get that z is the image of an element in $X^{(k)}$ (and thus in X) which is a simplicial cycle, and so the map $H_n^\Delta(X^{(k)}) \to H_n(X^{(k)})$ is surjevtive. Similarly, we can show that it is injective, which completes the proof.

1.11 Degree maps

Definition 1.31 (Degree of a map)**.** Let $f \colon S^n \to S^n$ be continuous. We then have that $f_* \colon H_n(S^n) \to H_n(S^n)$ is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ and as such, it is determined by $1 \mapsto k$. We call $\deg(f) := k$ the degree of f.

Proposition 1.35. *Degree maps satisfy the following properties:*

(1) $\deg(\mathrm{id}) = 1$.

(2) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

(3) If f is not surjective then $\deg(f) = 0$.

(4) If f is a reflection then $\deg(f) = -1$.

(5) Consider the antipodal map $f \colon S^n \to S^n$ given by $f(x) = -x$. Then $\deg(f) = \deg(-\mathrm{id}) = (-1)^{n+1}$.

(6) If $f \colon S^n \to S^n$ is a homeomorphism then $\deg(f) = \pm 1$.

(7) If $f \simeq g \colon S^n \to S^n$ then $\deg f = \deg g$.

(8) If $f \colon S^n \to S^n$ has no fixed points. Then $\deg(f) = (-1)^{n+1}$.

(9) Suppose n is an even natural number, and $G \curvearrowright S^n$ freely ($\forall x, g.x = x \implies g = 1$). Then $G \cong 1$ or $G \cong \mathbb{Z}_2$.

Proof.

(1) This follows immediately from the fact that $(\mathrm{id})_* = \mathrm{id}_{\mathbb{Z}}$.

(2) This follows immediately from the fact that $(f \circ g)_* = f_* \circ g_*$.

(3) Let $y \notin f(S^n)$. We can describe f as the composition of maps $S^n \xrightarrow{f} S^n - \{y\} \xrightarrow{i} S^n$ where the last is the inclusion map. Since $H_n(S^n - \{y\}) = \{0\}$ we must have that $f_* = 0$.

(4) We have seen that for $S^n = \Delta_1 \sqcup \Delta_2$ with the same orientatoin that $\widetilde{H}_n(S^n) = \langle \Delta_1 - \Delta_2 \rangle \cong \mathbb{Z}$. We have that $f(\Delta_1) = \Delta_2$ and $f(\Delta_2) = \Delta_1$ so $f_*([\Delta_1 - \Delta_2]) = -[\Delta_1 - \Delta_2]$.

(5) The antipodal map $f \colon S^n \to S^n$ maps $(x_1, x_2, \dots, x_{n+1})$ to $(-x_1, -x_2, \dots, -x_{n+1})$. This shows that it is a composition of $n+1$ reflections (one in each coordinate) and thus $\deg(f) = (-1)^{n+1}$.

(6) We can prove the more general claim that if f is a homotopy equivalence then $\deg(f) = \pm 1$. That is because

$$1 = \deg(\mathrm{id}) = \deg(f \circ g) = \deg(f) \cdot \deg(g)$$

and $1 = \deg(f) \cdot \deg(g)$ implies that $\deg(f) = \pm 1$.

(7) We know that if $f \simeq g$ then $f_* = g_*$ so $\deg(f) = \deg(g)$.

(8) We will show that f is homotopic to the antipodal map by the homotopy

$$H(x,t) = \frac{tf(x) + (1-t)(-x)}{\big\|tf(x) + (1-t)(-x)\big\|}.$$

Since $f(x) \neq x$ for all x, the denominator is not 0 for all $x \in S^n$ and $t \in [0,1]$, thus H is well defined and is a homotopy. The geometric reason that the denominator does not vanish, is because if $f(x) \neq x$, then the line segment connecting $f(x)$ and $-x$ does not pass through the origin.

(9) Recall that an action of a group G on a space X is a homomoprhism from G to the group of homeomorphisms $X \to X$. Since by (6) homeomorphisms $X \to X$ have degree $\{\pm 1\}$ the map $\deg \colon G \to \{\pm 1\}$ given by $\deg \colon g \mapsto \deg(g)$ is well defined. It is a homomorphism since $\deg(fg) = \deg(f)\deg(g)$. If the action is free, then by (8) the map deg sends each nontrivial element of G to $(-1)^{n+1}$. Thus when n is even $\ker(\deg) = \{1_G\}$ so deg is injective hence $G \subset \mathbb{Z}_2$.

□

Remark 1.50. We can use property (3) to prove the fundamental theorem of algebra. The proof can be found at the problem set.

Remark 1.51. The reverse direction of property (7) also holds. It is called the Hopf theorem, but we won't prove it

Theorem 1.36 (Hairy ball theorem)**.** *There does not exist a nonvanishing continuous vector field on S^n if and only if n is even. Equivalently, S^n has a continuous field of nonzero tangent vectors if and only if n is odd.*

Proof. Let $x \mapsto v(x)$ be a continuous field of nonzero tangent vectors on S^n. Assign to a vector $v\colon S^n \to \mathbb{R}^{n+1}$ the vector $v(x)$ tangent to S^n at x. We will show that n must be odd. Since $v(x) \neq 0$ for all $x \in S^n$ we can normalize so that $\|v(x)\| = 1$ for all x by replacing $v(x)$ with $v(x)/\|v(x)\|$. Similarly to how $\cos(t) + i\sin(t)$ lies on the unit circle on the complex plane, $(\cos t)x + \sin(t)v(x)$ lies on the unit circle on the plane spanned by x and $v(x)$ (this can be done because x and $v(x)$ are orthogonal). From this we obtain the homotopy map $H\colon S^n \times [0,1] \to S^n$ defined by

$$H(x,t) := \cos(\pi t)x + \sin(\pi t)v(x),$$

which a homotopy from the identity map to the antipodal map. This implies that

$$1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}.$$

It follows that n must be odd.

Suppose next that n is odd. Then $n = 2k-1$ for some $k \in \mathbb{N}$. We can define

$$v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) := (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}).$$

Then $v(x)$ is a evidently a continuous field of nonzero tangent vectors on S^n, which completes the proof. □

Definition 1.32 (Local degree)**.** Let $f\colon S^n \to S^n$ be a continuous map. Suppose that for some $y \in S^n$ the preimage $f(y)^{-1}$ is finite. Let $f^{-1}(y) = \{x_i\}_{i=1}^m$, let $\{U_i\}_{i=1}^m$ be disjoint neighbourhoods of each x_i and let V be the image of all of these neighbourhoods. Then $f(U_i - \{x_i\}) \subset V - \{y\}$ for all $1 \leq i \leq n$, and we have the diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V - \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

where the maps k_i and p_i are induced by inclusion, the map j is induced from the quotient map on the chain level, and f_* is the induced map of f. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Since $H_n(S^n) \cong \mathbb{Z}$ we obtain the the top two maps can be identified with $\mathbb{Z}$, and the map $(f_i)|_U\colon H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\})$ becomes a map of the form $1 \mapsto k_i$ for some $k_i \in \mathbb{Z}$. The local degree of f at x_i is set to be $\deg(f|_{x_i}) := k_i$.

Remark 1.52. Let $f\colon S^n \to S^n$ be a homeormophism. Then for any $x \in S^n$ there exists a unique $y \in S^n$ so that $f^{-1}(y) = \{x\}$ and then all the maps of the diagram from Definition 1.32 are isomorphisms and in particular $\deg(f|_x) = \pm 1$.

More generally, if $f\colon S^n \to S^n$ is a map that maps each neighbourhood U_i of x_i homeomorphically onto V, then $\deg(f|_x) = \pm 1$ for each i, and we can usually determine the sign as well.

The following proposition shows how we can use local degree to compute the global degree of a map.

Proposition 1.37. *Assume that $y \in S^n$ so that $f^{-1}(y) = \{x_1, \dots, x_m\}$. Then* $\deg(f) = \sum_{i=1}^m \left(f|_{x_i}\right)$.

Proof. Recall the diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[dl, "\cong"'] \arrow[r, "f_*"] \arrow[d, "k_i"] & H_n(V, V - \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

We notice that the central term $H_n(S^n, S^n - f^{-1}(y))$ is the direct sum of the groups $H_n(U_i, U_i - \{x_i\}) \cong \mathbb{Z}$, with k_i being the inclusion of the ith summand. The map p_i must be the projection onto the ith summand since the upper triangle commutes and $p_i \circ k_j = 0$ for $j \neq i$ as any element in the image of k_j is sent to zero in $H_n(S^n, S^n - \{x_i\})$ as a subset of $S^n - \{x_i\}$. Recall we can identify elements in the diagram as copies of $\mathbb{Z}$ and then commutativity in the lower triangle means that $(p_i \circ j)(1) = 1$, hence $j(1) = (1, 1, \dots, 1) = \sum_i k_i(1)$. Commutativity of the upper square implies that the middle f_* maps $k_i(1)$ to $\deg(f|_{x_i})$, hence $\sum_i k_i(1) = j(1)$ is mapped to $\sum_i \deg(f|_{x_i})$. Then since the lower square is commutative we get that $\deg(f) = \sum_i \left(f|_{x_i}\right)$ which completes the proof. ☐

Example 1.21. Let $f_k\colon S^1 \to S^1$ be defined by $f(z) = z^k$. We already know that $\deg(f_k) = k$ since we know that the generator of $H_1(S^1)$ (which by abuse of notation is the identity map viewed as a singular simplex) is sent to itself k times over. We will now show this fact by using local degree. When $k = 0$ we have $z \mapsto 0$ so f_0 is constant and since in particular it is not surjective so we get that $\deg(f_0) = 0$. Suppose that $z \mapsto z^{-1} = \bar{z}$. This map is a refelection so $\deg(f_{-1}) = -1$. Since $f_k = f_{-1} \circ f_k$, we can assume without loss of generality that $k > 0$. Pick $y = 1 \in S^1$ (we view S^1 as the circle in the complex plane). We have that $f_k^{-1}(y) = \{x_1, \dots, x_k\}$ where $x_j = e^{2\pi i j/k}$ for $1 \leq j \leq k$. Let $U_1, \dots, U_k$ be the disjoint neighbourhoods of each x_j, so that excluding x_i each U_i is mapped to $V = S^1 - \{1\}$. That is because the map $f|_{U_j}$ stretches the interval U_j by a factor of k, and rotates it to coincide with V. Since the degree is multiplicative, the degree of f at x_i is simply the degree of the rotation multiplied by the degree of the stretching, but since the stretching is homotopic to the identity, the degree of f at x_i is simply the degree of the rotation. Since a rotation is a homeomorphism, its local degree at any point is equal to its global degree, and since it is a rotation in a positive direction it's equal to 1. Thus,

$$\deg(f|_{x_j}) = \deg(r|_{x_j}) = \deg(r) = \deg(\mathrm{id}) = 1,$$

and we get that

$$\deg(f) = \sum_{j=1}^{k} \deg(f|_{x_j}) = k.$$

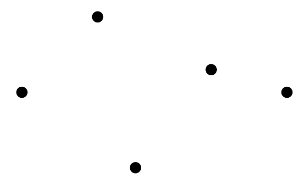
Remark 1.53. If we had a map $f\colon S^1 \to S^1$ that wraps a third of the circle around itself once, another third around itself twice, and the last third around itself in reverse direction four times the degree of f will be $\deg(f) = 1 + 2 - 4 = -1$.

1.12 Cellular complexes

Remark 1.54. In topology we can construct really terrible spaces. When we introduced a simple structure like the Δ-complex, it allowed us to compute the homology of a certain family of spaces. We now (re)introduce cell (CW) complexes, which generalize Δ-complexes and allows us to compute the homology of even more spaces, more easily.

Definition 1.33 (CW complex)**.** A CW (or cellular) complex is a space X built inductively as follows. First, we define the 0-skeleton which represents the 0-dimensional structure of the complex.

$X^{(0)}$ = A discrete set of vertices =

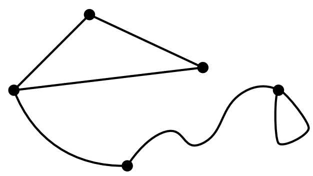


Then the 1-skeleton is defined by setting pairs of 1-discs and attaching maps $\{\mathbb{D}^1_\alpha, \varphi_\alpha\}$ where the attaching maps are $\varphi_\alpha \colon \partial\mathbb{D}^1_\alpha \to X^{(0)}$ so that

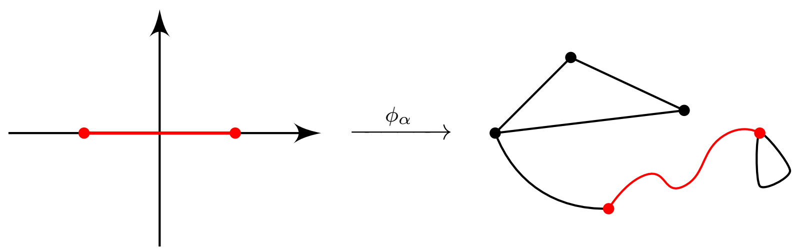
$$X^{(1)} = \left(X^{(0)} \sqcup \bigsqcup_\alpha \mathbb{D}^1_\alpha\right) / \left\{x \sim \varphi_\alpha(x)\right\}.$$

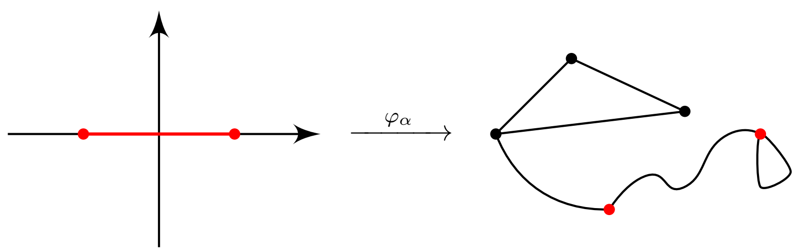
For example using the 0-skeleton from earlier we can have

$X^{(1)}$ =



We also have the characteristic map $\phi_\alpha \colon \mathbb{D}^1_\alpha \to X^{(1)}$ this maps sends each disc from "the void" to the disc in the context of the CW complex. The following diagram compares φ_α and ϕ_α.





Sometimes we call $\left\{\mathrm{Int}(\phi_\alpha(\mathbb{D}^1_\alpha))\right\}_\alpha$ the 1-cells of X.
We can now define a finite dimensional CW complex $X = X^{(n)}$ inductively as follows. Suppose we have a $X^{(n-1)}$-skeleton and pairs of n-discs and attaching maps $\left\{(\mathbb{D}^n_\alpha, \varphi_\alpha)\right\}_\alpha$. Then

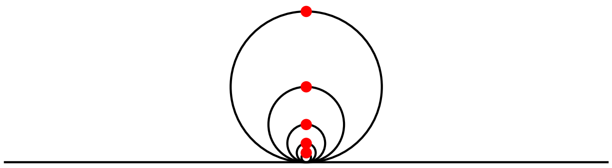
$$X^{(n)} = \left(X^{(n-1)} \sqcup \bigsqcup_\alpha \mathbb{D}^1_\alpha\right) / \left\{x \sim \varphi_\alpha(x)\right\}.$$

Remark 1.55. In 0 and 1 dimensions, every CW complex is equivalent to delta complex, and vice versa.

Remark 1.56. In a Δ-complex structure, the maps $\sigma_\alpha \colon \Delta^{n_\alpha} \to X$ were characteristic maps.

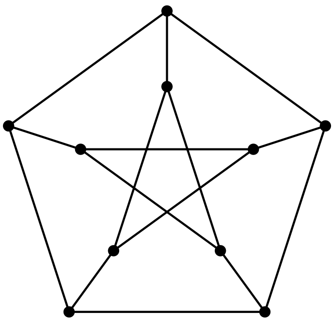
Remark 1.57. When $n \to \infty$ for some sequence $X^{(n)}$ we define $X := \lim_{n\to\infty} X^{(n)} = \bigcup X^{(n)}$ and say that $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open for all n.

Example 1.22. Before giving examples here is a standard nonexample. Consider $\mathbb{R}^2$ as the background, and add all circles with radius $1/n$ and center $(0, 1/n)$. We obtain a CW complex that looks like



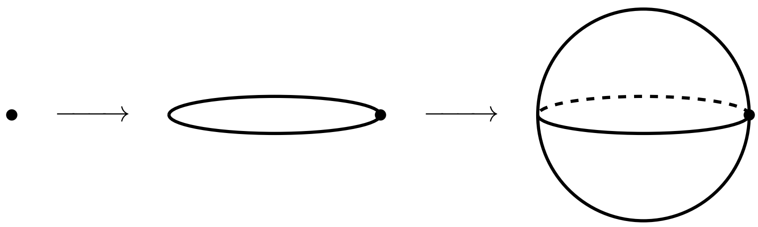
This is known as the *Hawaiian Earring*. It is not a CW complex because we required our vertices, the elements of $X^{(0)}$ to be discrete, and clearly the points $(0, 1/n)$ converge to $(0, 0)$. Counterintuitively, if we chose $(0, n)$ as the center for circle of radii n, we would get a well defined CW complex.

Example 1.23. A topological graph is a CW complex. For example



This is known as the *Petersen graph*. However, we must be careful. Recall that the Petersen graph is not a planar graph. This means that in the real plane, the 1-cells must intersect, which means this is not a CW complex, but we can embed this graph (as well as any other finite graph) in $\mathbb{R}^3$ without 1-cells intersections, so it does have a CW complex representation.

Example 1.24. We can define a cellular complex for S^2 by setting $X^{(0)} = \{v\}$ and letting $X^{(1)}$ be $X^{(0)} \sqcup \mathbb{D}^1/\varphi_1$ where $\varphi_1 \colon \partial\mathbb{D}^1 \to \{v\}$ is the constant map, and then gluing two more 2-discs by attaching maps that send their boundaries to $\phi_1(\mathbb{D}^1)$. We get something like



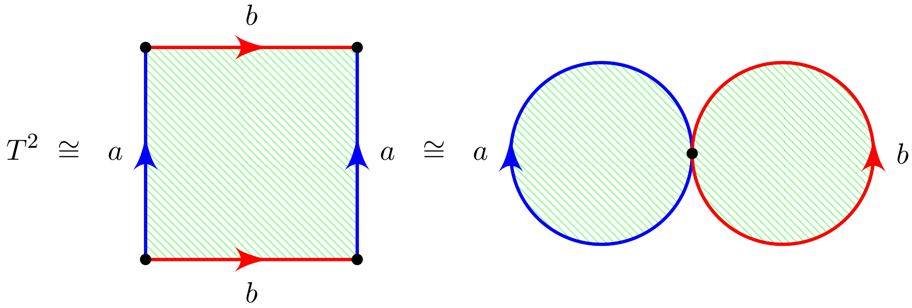
Note that this is not a Δ-complex.

Example 1.25. We can also construct a cellular complex for S^n by setting $X^{(0)} = \{x_0\}$ and $X^{(i)} = X^{(0)}$ for $1 \le i < k$, and then add a single n-cell by $(\mathbb{D}^n, \varphi)$ so that $\varphi \colon \partial\mathbb{D}^n \to X$ is the constant map $\varphi(x) = x_0$. Note that this is not a Δ-complex. The general construction looks like this:

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow n\text{-sphere with a point}$$

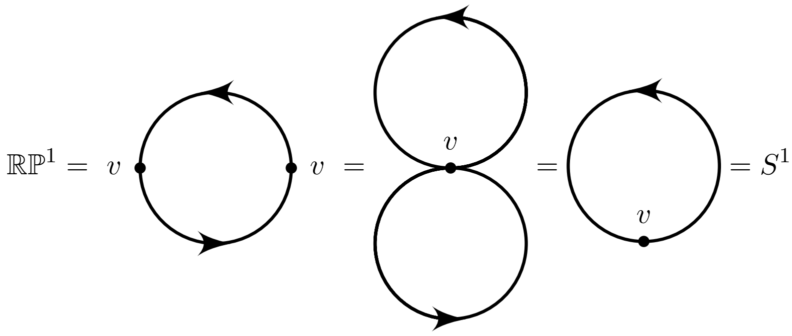
Remark 1.58. We can also construct a CW complex for S^2 with many many cells, so that it looks like a basketball, or a soccer ball. We can also simply triangulize S^2 as we did for a Δ-complex, however these methods will not prove very efficient when we will try to compute the homology of these complexes.

Example 1.26. We can construct a CW complex for the torus T^2 by noticing that

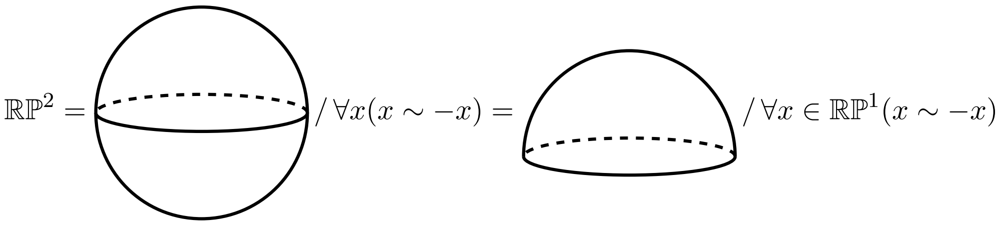


so we just need a single 0-cell denoted e^0, two 1-cells denoted e^1_a and e^1_b, with boundaries attached to e^0, and one 2-cell denoted e^2, the attaching map gluing its boundary along the word $aba^{-1}b^{-1}$, where the loops a and b represent the 1-cells e^1_a and e^1_b respectively.

Example 1.27. The space $\mathbb{RP}^n$ is just the space of lines in $\mathbb{R}^{n+1}$ passing through the origin $\mathbb{R}^{n+1} - \{0\}/x \sim \lambda x$. In particular, we get that $\mathbb{RP}^0$ is simply a point $\mathbb{D}^0 = \{*\}$. We also know that $\mathbb{RP}^n$ is homeomorohic to $S^n/x \sim -x$ so for $\mathbb{RP}^1$ we get that



so we have that $S^1 \cong \mathbb{RP}^1$ (but this is an anomaly). This shows that we can construct a CW complex for $\mathbb{RP}^1$ by attaching one 1-cell to one 0-cell in the obvious way. Equivalently, by attaching one 1-cell to $\mathbb{RP}^0$. When considering $\mathbb{RP}^2$, we get a 2-sphere with identified antipodal points. This is equal to the upper hemisphere of the 2-sphere, with the equator having its antipodal points glued together just like a copy of $\mathbb{RP}^1$. We will abuse notation and denote the equator of $\mathbb{RP}^2$ by $\mathbb{RP}^1$.



This shows that we can construct a CW complex for $\mathbb{RP}^2$ by attaching one 2-cell to $\mathbb{RP}^1$. This pattern continues, and in this way we can construct a CW complex for $\mathbb{RP}^n$ with one i-cell for all $0 \leq i \leq n$ for all n.

Remark 1.59. Formally, the pattern for the inductive construction of real projective spaces is $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \sqcup \bigsqcup \mathbb{D}^n / \{x \sim \varphi(x)\}$ with $\varphi_\alpha \colon \partial \mathbb{D}^n_\alpha \to \mathbb{RP}^{n-1}$ defined in the natural way.

Exercise 1.1. Think of a CW complex for $\mathbb{CP}^n$ (with a single cell in each even dimension). In particular, this implies that $\mathbb{CP}^1 \cong S^2$.

1.13 Cellular homology

Remark 1.60. For all subcomplexes $A \subseteq X$ the pair (X, A) is a good pair.

Lemma 1.38. *Let X be a CW complex so that $X = \bigcup X^{(n)}$.*

(1) We have that

$$H_k\left(X^{(n)}, X^{(n-1)}\right) = \begin{cases} \bigoplus_\alpha \mathbb{Z}[\phi_\alpha], & n = k, \\ 0, & n \neq k. \end{cases}$$

Note that ϕ_α are the characteristic maps.

(2) We have that $H_k(X^{(n)}) \cong \{0\}$ for $k > n$.

(3) We have that $H_k(X^{(n)}) \cong H_k(X)$ for $n > k$.

Proof.

(1) Follows from the fact that $(X^{(n)}, X^{(n-1)})$ is a good pair and that $X^{(n)}/X^{(n-1)}$ is a wedge of n-spheres.

(2) Consider this subsequence of the long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$:

$$(H_{k+1}X^{(n)}, X^{(n-1)}) \longrightarrow H_k(X^{(n-1)}) \longrightarrow H_k(X^{(n)}) \longrightarrow H_k(X^{(n)}, X^{(n-1)})$$

Since $k > n$ we get that $k \neq n$ and $k \neq n-1$ so

$$H_{k+1}(X^{(n)}, X^{(n-1)}) \cong H_k(X^{(n)}, X^{(n-1)}) \cong \{0\}\,.$$

By exactness of the sequence we get $H_k(X^{n-1}) \cong H_k(X^{(k)})$. Thus for $k > n$ we get

$$H_k(X^{(n)}) \cong H_k(X^{(n-1)}) \cong \cdots \cong H_k(X^{(0)}) = \{0\}\,,$$

which completes the second part of the lemma.

(3) For finite dimensional X ($X = X^{(n)}$ for some n), the statement follows from the same long exact sequence from (2).

□

Definition 1.34 (Cellular chains)**.** We define $C_n^{\mathrm{CW}} = H_n(X^{(n)}, X^{(n-1)})$ Note that this implies that $C_n^{\mathrm{CW}} \cong \bigoplus_\alpha \mathbb{Z}[\phi_\alpha]$ for characteristic maps ϕ_α of n-cells. The chain groups $C_n^{\mathrm{CW}}(W)$ will form our cellular chain complex.

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

Definition 1.35 (Cellular boundary map)**.** We want to construct the boundary maps d_n. We notice that we can combine the long exact sequences of the pairs (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}) to get a diagram of the form

$$\begin{tikzcd}
0 \arrow[dr] & & H_n(X^{n+1}) & & \\
& H_n(X^n) \arrow[ur] \arrow[dr, "q_n"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
& & H_{n-1}(X^{n-1}) \arrow[ur, "q_{n-1}"'] & & \\
& 0 \arrow[ur] & & &
\end{tikzcd}$$

We define $\partial^{\mathrm{CW}} := d_n = q_{n-1} \circ \partial_n$. Indeed we get $\left(\partial^{\mathrm{CW}}\right)^2 = q_* \circ \partial \circ q_* \circ \partial = 0$ because of exactness and $\partial \circ q_* = 0$.

Remark 1.61. We want a practical way to compute the boundary maps ∂_n^{CW}. Let $\varphi_\alpha \colon \partial\mathbb{D}_\alpha^n \to X^{(n-1)}$ be the attaching map of some n-cell. We then have the following sequence

$$S_\alpha^{n-1} \cong \partial\mathbb{D}_\alpha^n \xrightarrow{\varphi_\alpha} X^{(n-1)} \xrightarrow{q} X^{(n-1)}/X^{(n-2)} \cong \bigvee_\delta S_\delta^{n-1} \xrightarrow{p_\beta} \bigvee_\delta S_\delta^{n-1}/\bigvee_{\delta\neq\beta} S_\beta^{n-1} \cong S_\beta^{n-1}.$$

for each $(n-1)$-cell β. Define the map $\varphi_{\alpha\beta} \colon S_\alpha^n \to S_\beta^n$ to be the composition of these maps. Then we have

$$\partial^{\mathrm{CW}}\left([\phi_\alpha]\right) = \sum_\beta \deg(\varphi_{\alpha\beta}) \cdot [\phi_\beta].$$

Remark 1.62. Earlier we have used both d_n and ∂_n^{CW} interchangeably to denote the boundary map of the cellular chain complex. From now on we will only use d_n for brevity.

Remark 1.63. In order to compute the cellular homology of spaces, the only facts we have to remember are that $C_n^{\mathrm{CW}}(X) = \bigoplus_\alpha \mathbb{Z}[\phi_\alpha]$ where ϕ_α are characteristic maps of n-cells, and that the boundary maps are given by

$$d_n\left([\phi_\alpha]\right) = \sum_\beta \deg(\varphi_{\alpha\beta}) \cdot [\phi_\beta].$$

Example 1.28. Let $X = S^n$ for $n \geq 2$. As we saw in Example 1.25 we can construct a CW complex for X using only one 0-cell and one n-cell. The cellular chain of such a complex is of the form

$$\cdots \longrightarrow 0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} 0 \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

We don't even have to compute the boundary maps since they are all simply the zero maps, and we immediately get that

$$H_k^{\mathrm{CW}}(S^n) \cong \begin{cases} \mathbb{Z}, & k = 0 \text{ or } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

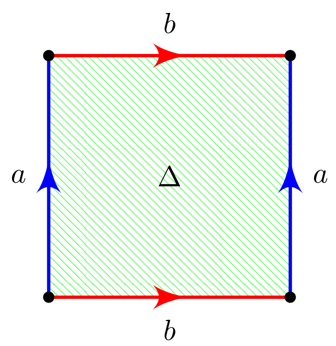
Example 1.29. Let $X = \mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/x \sim \lambda x$ for $0 \neq \lambda \in \mathbb{C}$. Then from Exercise 1.1 we know that X has a CW complex with one i-cell for each $i = 2k$ for $0 \leq k \leq n$. The cellular chain complex is

$$\cdots \longrightarrow 0 \xrightarrow{d_{2n+1}} \mathbb{Z} \xrightarrow{d_{2n}} 0 \xrightarrow{d_{2n-1}} \cdots \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

and the homology is

$$H_t^{\mathrm{CW}}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & t \leq 2n \text{ and } t \text{ even}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Example 1.30. Consider CW complex for the 2-torus we saw in Example 1.26



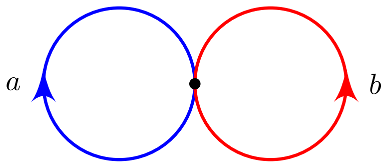
We can see that the cellular chain complex is of the form

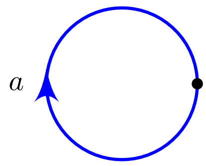
$$\cdots \xrightarrow{d_4} 0 \xrightarrow{d_3} \mathbb{Z}\Delta \xrightarrow{d_2} \mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow{d_1} \mathbb{Z}v \xrightarrow{d_0} 0$$

We know that CW complexes are the same as Δ-complexes in dimensions 0 and 1 so in particular $d_1(a) = v - v = 0$ and $d_1(b) = v - v = 0$. This means the only map which is (possibly) not the zero map in this chain is d_2. Using the formula for the boundary map we get

$$d_2(\Delta) = \deg(\varphi_{\Delta a})a + \deg(\varphi_{\Delta b})b.$$

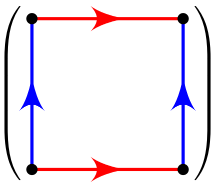
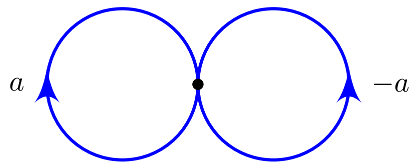
In order to compute these degrees we can notice that

$\varphi_{\Delta a} \colon \partial\mathbb{D}^2_\Delta \xrightarrow{\varphi_\Delta} X^{(1)} \cong$ 

$\xrightarrow{p_a}$ 

$= S^1_a$

Recall that $\varphi_{\Delta a}$ is a map between spheres, in particular $\varphi_{\Delta a} \colon S^1 \to S^1$, and we can see that it sends the boundary of $\mathbb{D}^2_\Delta$ to a map which is homotopy equivalent to the constant map.

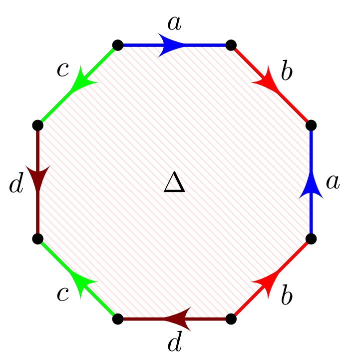
$\varphi_{\Delta a}$  $=$ 

$\simeq \bullet$

which, I admit, is a weird way to write that going over the boundary of the torus (which represents S^1) once, is mapped to going over S^1_a once in some direction, and then in the opposite direction, which, as we know is homotopy equivalent to the constant map. This means that $\varphi_{\Delta a} \equiv 0$ and similarly $\varphi_{\Delta b} \equiv 0$. Since all boundary maps are the zero maps, we get

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0 \text{ or } k = 2, \\ \mathbb{Z}^2, & k = 1, \\ \{0\}, & \text{otherwise.} \end{cases}$$

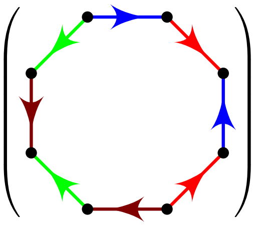
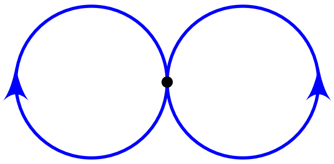
Example 1.31. Let $\Sigma_2 = T\#T$ be the compact orientable surface of genus two. I suggest looking for an online drawing of it, but here is its diagram as a quotient space



with all the vertices identified and denoted by v. The cellular chain complex is

$$\cdots \longrightarrow 0 \xrightarrow{d_3} \mathbb{Z}\Delta \xrightarrow{d_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}d \xrightarrow{d_1} \mathbb{Z}v \xrightarrow{d_0} 0$$

It is clear that all maps except (possibly) d_2 are zero maps. Similarly to what we saw in Example 1.30 we get that

$\varphi_{\Delta a}$  $= a$  $-a \simeq \bullet$

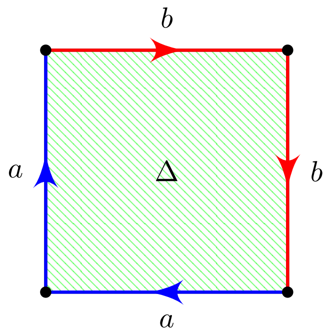
and so similarly $\varphi_{\Delta a}$, $\varphi_{\Delta b}$, $\varphi_{\Delta c}$ and $\varphi_{\Delta d}$ are all zero maps so $d_2 \equiv 0$ which means that

$$H_n(\Sigma_2) = \begin{cases} \mathbb{Z}, & n = 0 \text{ or } n = 2, \\ \mathbb{Z}^4, & n = 1 \\ \{0\}, & \text{otherwise.} \end{cases}$$

Remark 1.64. The argument we saw in Example 1.31 does not actually depend on the assumption that the genus of the surface is 2. Thus, we can generalize and say that in the case of the compact orientable surface of genus g, which is $\Sigma_g = \#^g T^2$, we have

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z}, & n = 0 \text{ or } n = 2, \\ \mathbb{Z}^{2g}, & n = 1 \\ \{0\}, & \text{otherwise.} \end{cases}$$

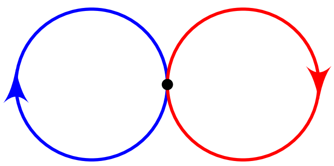
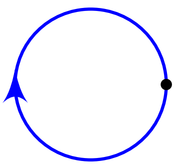
Example 1.32. Consider the Klein bottle K with the CW complex



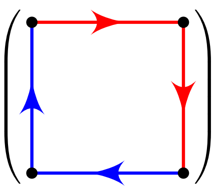
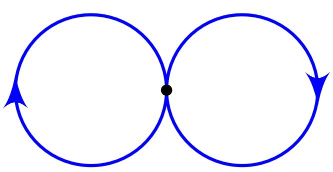
This is not the standard representation of the Klein bottle, but it is possible to get to the more standard representation but cutting along the anti diagonal and pasting along the side a. Using our representation we get a cellular chain complex of the form

$$\cdots \xrightarrow{d_4} 0 \xrightarrow{d_3} \mathbb{Z}\Delta \xrightarrow{d_2} \mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow{d_1} \mathbb{Z}v \xrightarrow{d_0} 0$$

As in all of the examples we have seen so far, the interesting map is d_2. This time, things are a little different. We have

$\varphi_{\Delta a} \colon \partial\mathbb{D}^2_\Delta \xrightarrow{\varphi_\Delta} X^{(1)} \cong a$  $b \xrightarrow{p_a} a$  $= S^1_a$

and

$\varphi_{\Delta a}$  $= a$  $a \simeq z^2$

so $\varphi_{\Delta a}$ sends the boundary of the Klein bottle (one trip around $S^1 \cong \partial\mathbb{D}^2_\Delta$) to two trips around S^1_a (just like z^2) and so $\deg(\varphi_{\Delta a}) = 2$. Similarly we get that $\deg(\varphi_{\Delta a}) = 2$, and thus we have

$$d_2(\Delta) = 2(a+b).$$

Therefore the only nontrivial homology we have to calculate is

$$H_1(K) = \frac{\ker d_1}{\operatorname{im} d_2} = \frac{\mathbb{Z}a \oplus \mathbb{Z}b}{\mathbb{Z}(2(a+b))} = \langle a, a+b \mid 2(a+b)\rangle = \langle a\rangle \oplus \langle a+b \mid 2(a+b)\rangle \cong Z \oplus \mathbb{Z}/2\mathbb{Z}.$$

Therefore

$$H_n(K) = \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}, & n = 1, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Example 1.33. Let $X = \mathbb{RP}^n$. We know that X has a CW complex with one i-cell in each dimension $0 \le i \le n$. Thus the cellular chain complex is

$$\cdots \xrightarrow{d_{n+2}} 0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

It is not so clear what the boundary maps are anymore. Recall that

$$S^{n-1}_\alpha \cong \partial\mathbb{D}^n_\alpha \xrightarrow{\varphi_\alpha} X^{(n-1)} \xrightarrow{q} X^{(n-1)}/X^{(n-2)} \cong \bigvee_\delta S^{n-1}_\delta \xrightarrow{p_\beta} \bigvee_\delta S^{n-1}_\delta / \bigvee_{\delta\neq\beta} S^{n-1}_\beta \cong S^{n-1}_\beta.$$

so in our case since $X^{(n)} = \mathbb{RP}^n$ and there's only one cell in each dimension we get

$$\varphi_{\Delta_n\Delta_{n-1}} \colon S^{n-1} \cong \partial\mathbb{D}^n \xrightarrow{\varphi_{\Delta_n}} \mathbb{RP}^{n-1} \xrightarrow{q} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1}$$

and

$$d_n\left([\phi_{\Delta_n}]\right) = \sum_\beta \deg(\varphi_{\Delta_n\Delta_{n-1}}) \cdot [\phi_{\Delta_{n-1}}] = \deg(\varphi_{\Delta_n\Delta_{n-1}}) \cdot [\phi_{\Delta_{n-1}}],$$

where Δ_n represents the n-cell of the complex. Note that the attaching map $\varphi_{\Delta_n} \colon S^{n-1} \to S^{n-1}/x \sim -x \cong \mathbb{RP}^{n-1}$ is given by the canonical double cover of $\mathbb{RP}^{n-1}$. We now use Theorem 1.37. Let $y \in S^{n-1}$ not on the equator, then $\varphi^{-1}_{\Delta_n\Delta_{n-1}}(y) = \{x, -x\}$ for some $x \in S^1$. This means that

$$\deg(\varphi_{\Delta_n\Delta_{n-1}}) = \deg(\varphi_{\Delta_n\Delta_{n-1}}|_x) + \deg(\varphi_{\Delta_n\Delta_{n-1}}|_{-x}).$$

However from the multiplicativity of the degree $\deg(\varphi_{\Delta_n\Delta_{n-1}}|_{-x}) = \deg(\varphi_{\Delta_n\Delta_{n-1}}|_x)\cdot\deg(\alpha|_{-x})$ where α is the antipodal map. Since we know that $\varphi_{\Delta_n\Delta_{n-1}}$ is homotopy equivalent to the identity in a sufficiently small neighbourhood of x, and that $\deg(\alpha|_{-x}) = (-1)^n$. It follows that $\deg(d_n) = 2$ when n is even and $\deg(d_n) = 0$ when n is odd. This implies that when n is odd

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & k = 0 \text{ and } k = n, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < n \text{ and } k \text{ is odd}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

and when n is even

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < k < n \text{ and } k \text{ is odd}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

1.14 Extras before cohomology

Definition 1.36 (Euler characteristic)**.** Let X be a CW complex with finitely many cells. Then we define the Euler characterstic of X to be

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \# \{i\text{-cells in } X\} = \#\text{Vertices} - \#\text{Edges} + \#\text{2-faces} - \cdots$$

Theorem 1.39. *The number $\chi(X)$ is independent of the CW complex on X. Moreover, if $X \simeq Y$ where X and Y are finite CW complexes, then $\chi(X) = \chi(Y)$.*

Definition 1.37 (Rank)**.** Let A be a finitely generated abelian group. Then from the classification of finitely generated abelian groups

$$A = \mathbb{Z}^r \oplus B$$

for some finite group B and some $r \geq 0$. We define $\operatorname{rank}(A) = r$.

Exercise 1.2. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups. Then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

Theorem 1.40. *Let X be a finite CW complex. Then*

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rank}\big(H_i(X)\big)$$

where $H_i(X)$ are finitely generated because C_i^{CW} is finitely generated.

Proof. We have the chain complex

$$\cdots \longrightarrow C_{i+1}^{\text{CW}} \xrightarrow{d_{i+1}} C_i^{\text{CW}} \xrightarrow{d_i} C_{i-1}^{\text{CW}} \longrightarrow \cdots$$

so in general if we denote $B_n = \operatorname{im} d_{n+1}$ and $Z_n = \ker d_n$ so that $H_n = Z_n/B_n$ we get the short exact sequences

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i^{\text{CW}}(X) \longrightarrow 0$$

and

$$0 \longrightarrow Z_i \longrightarrow C_i^{\text{CW}}(X) \longrightarrow B_{i-1} \longrightarrow 0$$

so from Exercise 1.2 we obtain the formulas

$$\begin{aligned} \operatorname{rank}(Z_i) &= \operatorname{rank}(B_i) + \operatorname{rank}(H_i) \\ \operatorname{rank}(C_i^{\text{CW}}) &= \operatorname{rank}(B_{i-1}) + \operatorname{rank}(Z_i) \end{aligned}$$

so by simple arithmetic calculations

$$\begin{aligned} \chi(X) &= \sum_{i=0}^{\infty}(-1)^i \# \{i\text{-cells in } X\} \\ &= \sum_{i=0}^{\infty}(-1)^i \big(\operatorname{rank}(B_{i-1}) + \operatorname{rank}(Z_i)\big) \\ &= \sum_{i=0}^{\infty}(-1)^i \big(\operatorname{rank}(Z_{i-1}) - \operatorname{rank}(H_{i-1}) + \operatorname{rank}(Z_i)\big) \\ &= \sum_{i=0}^{\infty}(-1)^{i-1} \operatorname{rank}\big(H_{i-1}(X)\big) \end{aligned}$$

where the last step follows from the fact the series is a telescopic sum. □

Remark 1.65. In every odd dimension the Euler characteristic of any compact manifold is zero.

Theorem 1.41 (Jordan curve theorem)**.** *Let $h\colon S^1 \to S^2$. Then $h(S^1)$ decomposes S^2 into two path connected components. Formally, $S^2 - h(S^1)$ has two path connected components, or in the language of homology, $\widetilde{H}_0(S^2 - h(S^1)) = \mathbb{Z}$.*

Theorem 1.42 (Jordan–Schoenflies)**.** *Let $h\colon S^1 \to S^2$. Then h can be extended to a homeomorphism $H\colon S^2 \xrightarrow{\cong} S^2$ so that $H|_E = h$, where $E \cong S^1$ is the equator of S^2.*

Example 1.34. This theorem can not be generalized to higher dimensions. Let $h(S^2) \subset S^3$ be Alexander's horned sphere. It's a counterexample, it's very hard to draw here just look it up.

Theorem 1.43.

(1) Let $h\colon \mathbb{D}^k \hookrightarrow S^n$ be a continuous map. Then $\widetilde{H}_i(S^n - h(\mathbb{D}^k)) = 0$ for all i.

(2) Let $h\colon S^k \hookrightarrow S^n$ be a continuous map so that $k < n$. Then

$$\widetilde{H}_i(S^n - h(S^k)) \cong \begin{cases} \mathbb{Z}, & i = n-k-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof.

(1) We prove this by induction on k. When $k = 0$ we have $\mathbb{D}^k = \{*\}$ and then $S^n - \{*\} \cong \mathbb{R}^n$ so since this is contractible $\widetilde{H}_i(S^n - h(\mathbb{D}^k)) = \widetilde{H}_i(\mathbb{R}^n) = 0$ for all i.

Next we identify $\mathbb{D}^k \cong [0,1]^k = I^k$. Define

$$A' = [0, 1/2] \times I^{k-1}$$
$$B' = [1/2, 1] \times I^{k-1}$$

Then we define $A = S^n - h(A')$ and $B = S^n - h(B')$ which are open. We then have $A \cap B = S^n - h(I^k)$ which means we want to compute $\widetilde{H}_i(A \cap B)$. In order to use Mayer–Vietoris we compute the union $A \cup B = S^n - h(A \cap B) = S^n - \{1/2\} \times I^{k-1}$. Then from Mayer–Vietoris we get

$$\cdots \longrightarrow \widetilde{H}_{i+1}(A \cup B) \longrightarrow \widetilde{H}_i(A \cap B) \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow \widetilde{H}_i(A \cup B) \longrightarrow \cdots$$

but since the groups at the edges are trivial we have $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$. Assume, by way of contradiction, that $\widetilde{H}_i(A \cap B)$ is not trivial. Then there exists $0 \neq c \in \widetilde{H}_i(A \cap B)$ so that either $0 \neq c \in \widetilde{H}_i(A)$ or $0 \neq c \in \widetilde{H}_i(B)$. In the first case denote $A_1 = A$, in the second $A_1 = B$. Continue the process on A_1 instead of S^n and so on indefinitely to get a sequence $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ so that $A_1' \supseteq \cdots \supseteq A_n' \supseteq$ indefinitely. Then we have $\bigcap A_n' = \{*\} \times I^{k-1}$. This implies that $0 \neq c \in \widetilde{H}_i(S^n - h(\{*\} \times I^{k-1})) = 0$, but by the induction hypothesis we get that $c = \partial d$ for some $d \in C_{i+1}(S^n - h(\{*\} \times I^{k-1}))$. to be finished.

(2) We know that $S^k = D_1 \sqcup D_2$ for some k-discs glued at their boundary. Then again by letting

$$A = S^n - h(D_1)$$
$$B = S^n - h(D_2)$$

we get from Mayer–Vietrois and part (1) the desired isomorphism.

□

Definition 1.38 (Wild knot)**.** An embedding $h\colon S^1 \hookrightarrow S^3$ is called a wild knot.

Remark 1.66. A wild knot is a called "wild" because it can be very pathological. A knot that is not wild is called tame. One way to rid the knot of pathological behaviour, is by requiring it to be smooth. Smooth knots are not the only tame knots.

Corollary 1.44. *Let h be a wild knot. Then $\widetilde{H}_i(S^3 - h(S^1)) = \mathbb{Z}$.*

Remark 1.67. In modern terminology a domain U is an open connected set. A long time ago a domain was just another term for an open set.

Theorem 1.45 (Invariance of domain)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}^n$ be a continuous and injective map. Then $f(U) \subseteq \mathbb{R}^n$ is open. It follows that f is an open map.*

Proof. We know that $\mathbb{R}^n \subset S^n$. We will show that $f\colon U \to S^n$ is open. Let $x \in \mathbb{D} \subset U$. We have $\mathbb{D} = \mathbb{D}^n \subseteq \mathbb{R}^n$. From the previous Jordan theorem we know that the $n-(n-1)-1 = 0$th homology is

$$\widetilde{H}_0(S^n - f(\partial\mathbb{D})) = \mathbb{Z}.$$

This means that $S^n - f(\partial\mathbb{D})$ has two path connected components. We know that the path connected components are

$$\begin{cases} f(\mathbb{D} - \partial\mathbb{D}) \\ S^n - f(\mathbb{D}) \end{cases}$$

Since path connected components of a locally path connected space are open, thus in particular $f(\mathbb{D} - \partial\mathbb{D}) = f(\mathring{\mathbb{D}})$ is open. □

Definition 1.39 (Topological n-manifold)**.** Let M be a second countable Hausdorff space. Then M is said to be an n-manifold if M is locally homeormophic to $\mathbb{R}^n$. Moreover, M is a closed n-manifold if M is compact.

Theorem 1.46. *Let N and M be closed connected n-manifolds and let $f\colon M \to N$ be a continuous injective map. Then f is a homeomorphism.*

Proof. From Theorem 1.45 f is open. Thus $f(M)$ is open and closed. Since N is connected, it follows that $f(M) = N$. □

1.15 Homology with coefficients

Remark 1.68. $\mathbb{Z}$ is the most important group in the world.

Remark 1.69. Let $(G, +)$ be an abelian group. Then we can define $C_n(X; G) = \bigoplus G\sigma$. This chain complex induces homology groups $H_n(X; G)$.

Remark 1.70. All previous propositions and theorems like excision, Mayer–Vietoris still apply when doing homology with coefficients.

Remark 1.71. Let $f\colon S^n \to S^n$ be a map. We then have $f_*\colon H_n(S^n; G) \to H_n(S^n; G)$ and since $H_n(S^n; G) \cong G$ we have $f_*\colon G \to G$ and in particular $f_*(g) = \deg(f) \cdot g$.

Example 1.35. Let $X = \mathbb{RP}^n$ and $G = \mathbb{Z}/2\mathbb{Z}$. Recall example Example 1.33. Now we have a chain complex of the form

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and in particular

$$H_i(\mathbb{RP}^n : \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & 0 \leq i \leq n, \\ 0, & i > n. \end{cases}$$

Theorem 1.47. *Let $f\colon S^n \to S^n$ be an odd function. Then* $\deg(f)$ *is odd.*

Proof. We know from the previous topology course that for the quotient map $g\colon S^n \to \mathbb{RP}^n = S^n/x \sim -x$ we have that for all $\sigma\colon \Delta^n \to \mathbb{RP}^n$ continuous that there exist only two lifts $\tilde{\sigma}_1, \tilde{\sigma}_2\colon \Delta^k \to S^k$ such that $q \circ \tilde{\sigma}_i = \sigma$. We also have that $\tilde{\sigma}_2 = \alpha \circ \tilde{\sigma}_1$ where $\alpha\colon S^n \to S^n$ is $\alpha(x) = -x$.

Denote $P^n = \mathbb{RP}^n$. We have the short exact sequence

$$0 \longrightarrow C(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \longrightarrow C(S^n;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{f_\#} C(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

This induces a long eact sequence in homology

$$H_{n+1}(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_n(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_n(S^n;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_n(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cdots} \bullet$$

Because of known results $H_0(S^n;\mathbb{Z}/2\mathbb{Z}) \cong H_n(S^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$H_i(\mathbb{RP}^n : \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & 0 \le i \le n, \\ 0, & i > n. \end{cases}$$

we get a few isomorphisms $H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong H_{k-1}(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ for almost all k.

We have the commutative diagram

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "g"'] & S^n \arrow[d, "g"] \\ \mathbb{RP}^n \arrow[r, "\bar{f}"] & \mathbb{RP}^n \end{tikzcd}$$

and a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\bar{f}_\#"] & C(S^n;\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\bar{f}_\#"] & C(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\bar{f}_\#"] & 0 \\ 0 \arrow[r] & C(\mathbb{RP}^n) \arrow[r] & C(S^n) \arrow[r] & C(\mathbb{RP}^n) \arrow[r] & 0 \end{tikzcd}$$

This does not immediately commute but we can verify that it commutes and is exact. Thus f_* induces isomorphisms $f_*\colon H_n(S^n, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_n(S^n;\mathbb{Z}/2\mathbb{Z})$. Thus f_* is a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ so it must be that $\deg(f) = 1 \pmod 2$. ☐

Theorem 1.48 (Borsuk–Ulam)**.** *Let $S^n \to \mathbb{R}^n$ be a continuous function. Then there exists $x \in S^n$ so that $f(x) = f(-x)$.*

Proof. Assume, by way of contradiction, there does not exist $x \in S^n$ so that $f(x) = f(-x)$. We can define $g\colon S^n \to \mathbb{R}^n - \{0\}$ so that $g(x) = f(x) - f(-x)$. Let $\tilde{g} := g(x)/\|g(x)\|$. In particular $\tilde{g}\colon S^n \to S^{n-1}$ and $\tilde{g}$ is odd.

From Theorem 1.47 the function g is odd ☐

2 Cohomology

Definition 2.1. Let G be an abelian group. Then we define

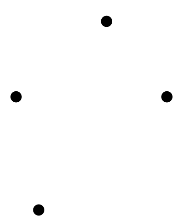
$$C^n(X;G) = \operatorname{Hom}(C_n(X;\mathbb{Z}), G)$$

where $\operatorname{Hom}(C_n(X), G)$ denotes the homomorphisms of abelian groups. Recall that $C_n(X;\mathbb{Z}) \cong \bigoplus_{\sigma\in I} \mathbb{Z}\cdot\sigma$. Equivalently,

$$C^n(X;G) = \left\{f\colon\ \{\sigma\}_{\sigma\in I} \to G\right\}.$$

called cochains or 0-cochains

Example 2.1. Let $G = \mathbb{Z}$. Suppose we had a graph



We had that $C_0 = \bigoplus \mathbb{Z}\sigma$ and now we have $\prod \mathbb{Z}\sigma$. The boundary map is $C^1 \xleftarrow{\delta} C^0$. It is given by

$$\delta f(e) = f(e_+) - f(e_-).$$
$$f\colon V \to \mathbb{Z}$$
$$\delta f\colon E \to \mathbb{Z}$$

where e_+ and e_- are the vertices of the directed edges.

Example 2.2. Given a graph with weights on the vertices. Can we construct a kduma function so that defined like in f its image will math the weights? If we can, we can always add $+C$ like in normal analysis

Remark 2.1. For g to have a primitive $f\colon g = \delta f$. We need $\delta g = 0$.

Definition 2.2 (Cohomology)**.** Let G be an abelian group. Let X be a topological space with a CW/Δ-complex structure on it that gives a chain complex

$$\cdots \longrightarrow C_{i+1}(X) \xrightarrow{\partial} C_i(X) \xrightarrow{\partial} C_{i-1}(X) \longrightarrow \cdots$$

We define

$$C^i(X;G) = \operatorname{Hom}(C_i(X), G).$$

We define $\delta\colon C^i(X;G) \to C^{i+1}(X;G)$.

$$\delta := \partial^*$$

so that

$$\delta f = f \circ \partial.$$

Remark 2.2. Let $f \in C^0(X;G)$. We can think of f as a map from the 0-simplices to G.

We see that

$$\delta f(e) = \partial^* f(e) = f\circ\partial e = f(e_+ - e_-) = f(e_+) - f(e_-).$$

When we have $\delta f \equiv 0$ we get that for every path e that $f(e_+) = f(e_-)$. This happens if and only if f is constant on path components of X. Therefore

$$H^0(X;G) = \ker\delta = \left\{f\colon X \to G \mid f \text{ is constant on path connected components}\right\}$$

Compare to $\bigoplus_{\text{text connected components}} \mathbb{Z}$. The direct sum is the dual of the direct product.

$$H^0(X;G) = \big(H_0(X;G)\big)^*.$$

We may conjecture that

$$H^n(X;G) = \big(H_n(X;G)\big)^*.$$

This is not far from the truth, but it is not the truth.

Example 2.3. We compute $H^i(\mathbb{RP}^3)$. Recall the the cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \longrightarrow 0$$

Since $G = \mathbb{Z}$ the dual chain complex is

$$0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \xleftarrow[2]{} \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \longleftarrow 0$$

We have $\alpha\colon 1 \mapsto m$. And $\alpha^*\colon ?$

The torsion $\mathbb{Z}/2\mathbb{Z}$ moved up one dimension.

Remark 2.3. This is not far from the truth, but it is not the truth. We have now seen this ni the example

2.1 Universal coefficient theorem

Remark 2.4. The goal is to describe $H^i(X;G)$ in terms of $H_i(X)$ and G.

Remark 2.5. The first goal is to construct a map $H^i(X;G)$ to $\big(H_i(X)\big)^*$.

Let $f \in \ker\delta = \mathbb{Z}^i(X;G)$ (an i-cocycle). We have that

$$f\partial = \delta f = 0 \iff f|_{B_i(X)} \equiv 0.$$

Then $f\colon C_i(\mathcal{C}) \to G$ induces a map $f|_{Z_i}\colon Z_i \to G$ which induces $\bar{f}\colon Z_i/B_i = H_i(\mathcal{C}) \to G$.

If $f \in B^i = \operatorname{im}\delta$ then

$$f = \delta\psi \implies f = \psi \circ \partial \implies f|_{Z_i}(z) = \psi\,\underbrace{\partial(z)}_{=0} \implies f|_{Z_i} \equiv 0$$

This implies that $\bar{f} \equiv 0$. Therefore the map

$$h\colon H^i(X;G) \to \big(H_i(X)\big)^*$$
$$h(f) = \bar{f}$$

completes our first goal.

Remark 2.6. Show the surjectivity in the example

Lemma 2.1 (Splitting lemma)**.** *Assume we have the short exact sequence of abelian groups*

$$0 \longrightarrow A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \longrightarrow 0$$

Then the following are equivalent

(1) There exists $s\colon C \to B$ so that $\beta \circ s = \mathrm{id}_C$.

(2) There exists a "retraction" $r\colon B \to A$ sp that $\gamma \circ \alpha = \mathrm{id}_A$.

(3) $B \cong A \oplus C$ such that α and β are the obvious map.

Remark 2.7. This equivalency is equivalent to the axiom of choice.

Proof. The directions $(3 \to 1)$ and $(3 \to 2)$ are obvious, so we will only show directions $(1 \to 2)$ and $(2 \to 3)$.

$(1 \to 2)$ Let $b \in B$, then $\beta(s\beta b - b) = 0$, so by exactness $(s\beta b - b) \in \ker(\beta) = \operatorname{im}(\alpha)$. Since α is injective there exists a unique $a \in A$ such that $\alpha(a) = s\beta b - b$. We define $r \colon B \to A$ by $r(b) := a$. In this way we have $r\alpha(a) = a$ so $r\alpha = \mathrm{id}_A$ and it is left to show that r is a homomorphism. Denote $r(b_1 + b_2) = a$, $r(b_1) = a_1$ and $r(b_2) = a_2$. We now have from previous definitions that

$$\alpha(a - (a_1 + a_2)) = s\beta(b_1 + b_2) - (b_1 + b_2) - (s\beta(b_1 + b_2) - (b_1 + b_2)) = 0,$$

but α is injective so $a - (a_1 + a_2) = 0$, which means that $a = a_1 + a_2$, which completes this part of the proof.

$(2 \to 3)$ Define $\varphi \colon B \to C \oplus A$ by $\varphi(b) = (\beta(b), r(b))$. The map φ is a homomorphism, and we will show that it is an isomorphism directly. If $b \in \ker(\varphi)$ then $\beta(b) = r(b) = 0$, but since $b \in \ker(\beta) = \operatorname{im}(\alpha)$, there exists a so that $\alpha a = b$, so $r(\alpha a) = r(b) = 0$, but also by the assumption $r(\alpha a) = \mathrm{id}_A(a) = a$. This implies that $a = 0$, and thus $b = 0$, so $\ker(\varphi) = 0$, and φ is an injection. We will now show surjectivity. Let $(c, a) \in C \oplus A$. Since β is surjective, there exists $b \in B$ so that $\beta(b) = c$. Since $\operatorname{im}\alpha = \ker\beta$ and $\alpha(a - r(b)) \in \operatorname{im}(\alpha)$ we get $\beta(b + \alpha(a - r(b))) = \beta(b) = c$, and we also have $r(b + \alpha(a - r(b))) = r(b) + (a - r(b)) = a$, so $\varphi(b + \alpha(a - r(b))) = (c, a)$, which means it is a surjection. We have shown that φ satisfies all properties of an isomorphism, therefore it is an isomorphism.

$$\begin{tikzcd}
 & & B \arrow[dd, "\varphi"] \arrow[dr, "\beta"] & & \\
0 \arrow[r, "0"] & A \arrow[ur, "\alpha"] \arrow[dr, "i"'] & & C \arrow[r, "0"] & 0 \\
 & & A \oplus C \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

We have shown that φ is an isomorphism. Since $\varphi\alpha(a) = r\alpha a = i_A(a)$ for all a, we get that α is the inclusion map. Since $\pi\varphi(b) = \beta(b)$, it is clear that the projection π is β (equivalently β is the projection), which completes the proof.

□

Definition 2.3 (Split sequence)**.** If a short exact sequence satisfies any of the conditions above, we say that it splits, or that it is a split sequence.

Remark 2.8. We show that h is surjective.
Under the assumption that C_i are abelian groups ..

Remark 2.9. We show that h depends only on $H_*(\mathcal{C})$ and G.

We define $H_i(F^*; G) = \mathrm{Ext}^i(H; G)$.

Theorem 2.2. *Let $\mathcal{C}$ be a free chain complex. Then the following short exact sequence splits*

$$0 \longrightarrow \mathrm{Ext}^1(H_{i-1}(\mathcal{C}), G) \longrightarrow H^i(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_i(\mathcal{C}); G) \longrightarrow 0$$

Theorem 2.3. *Let $\mathcal{C}$ be a chain complex of finitely generated abelian groups. Then the following short exact sequence splits.*

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(\mathcal{C}), G) \longrightarrow H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}); G) \longrightarrow 0$$

Proof. Recall that

$$\mathrm{Ext}(H_1 \oplus H_2; G) = \mathrm{Ext}(H_1; G) \oplus \mathrm{Ext}(H_1; G).$$

Since

$$\mathrm{Ext}(\mathbb{Z}; G) = 0$$
$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}; G) = G/nG$$

□

Remark 2.10. Let $H = F \oplus T$ be a finitely generated abelian group and $G = \mathbb{Z}$. Then

$$\mathrm{Ext}(H, \mathbb{Z}) = T = \text{torsion of } H.$$

Remark 2.11. Consider $\mathrm{Hom}(H, \mathbb{Z}) = F^* \cong F$.

Remark 2.12. The following corollary is a very good way to compute cohomology, under the assumption we know the homology groups.

Corollary 2.4. *Let X be a topological space so that $H_n(X)$ is finitely generated. Then*

$$H^n(X; \mathbb{Z}) \cong \textit{torsion of } H_{n-1}(X) \oplus \textit{free part of } H_n(X).$$

Remark 2.13. This is a good time to compute the cohomology of $\mathbb{RP}^3$ using the homology of $\mathbb{RP}^3$, and notice that it turns out the same as when we computed the cohomology of $\mathbb{RP}^3$ directly.

Remark 2.14 (Naturality)**.** Let $\mathcal{C} \xrightarrow{\alpha} \mathcal{C}'$ be two chain complexes connected by a map α. Suppose you also have $C'^* \xrightarrow{\alpha^*} C^*$. Then the following diagram commutes

Remark 2.15. Let $\mathbb{F}$ be a field. We have

$$H^n(\mathcal{C}; \mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(H_n(\mathcal{C}; F), \mathbb{F}).$$

Remark 2.16. Let X be a topological space. We have that

$$H^0(X; G) = \underbrace{\mathrm{Ext}(H_{-1}(X), G)}_{=0} \oplus \mathrm{Hom}(H_0(X), G) = \mathrm{Hom}(H_0(X), G).$$

We also have that

$$H^1(X; G) = \underbrace{\mathrm{Ext}(H_0(X), G)}_{=0} \oplus \mathrm{Hom}(H_1(X), G) = \mathrm{Hom}(H_1(X), G).$$

This is because H_0 is always free. Moreover,

$$H^1(X; G) = \mathrm{Hom}(H_1(X), G) = \mathrm{Hom}(\pi_1(X), G),$$

since $H_1(X)$ is the abelianization of $\pi_1(X)$.

Remark 2.17. Category theory arose from algebraic topology.

Remark 2.18. Homotopy invariance still holds for cohomology. Let $f, g \colon X \to Y$ be so that $f \simeq g$. Then $f^* = g^* \colon H^n(X) \to H^n(Y)$.

Remark 2.19. Relative cohomology and cohomology of a pair is the same.

Remark 2.20. In excision we had that if $Z \subseteq A \subseteq X$ so that $\overline{Z} \subseteq \mathring{A}$. Then

$$H_n(X - Z, A - Z) \xrightarrow{\cong} H_n(X, A).$$

Similalry, we have

$$H^n(X - Z, A - Z) \xrightarrow{\cong} H^n(X, A).$$

However, $H_n(X - Z, A - Z)$ and $H^n(X - Z, A - Z)$ are not always dual of each other.

2.2 Cup product

Definition 2.4 (Cup product)**.** The cup product is an operator

$$\cup\colon H^k(X) \times H^l(X) \to H^{k+l}(X).$$

Equivalently

$$\cup\colon C^k(X) \times C^l(X) \to C^{k+l}(X).$$

Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$. We map $\sigma\colon \Delta^{k+l} \to X$ as follows:

$$\varphi \cup \psi(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,v_{k+l}]}\right)$$

The definition is multiplication is not necessarily defined. Recall that φ and ψ map to an abelian group G, which only has addition. Thus the cup product is defined if and only G is a ring like $\mathbb{Z}$ or a field $\mathbb{F}$.

Remark 2.21. The cup product $\cup$ is blinear. This exactly means that the multiplication is distributive.

Lemma 2.5. *Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$. We have that*

$$(()\,\varphi \cup \psi) = ((\delta\varphi) \cup \psi) + (-1)^k\left(\varphi \cup (\delta\psi)\right).$$

Proof. Take $\sigma\colon [v_0, \dots, v_{k+l+1}] \to X$. We have that □

Remark 2.22. Let φ and ψ be cocycles. Then $\varphi \cup \psi$ is a cocycle.

Example 2.4. Let Σ_g be the orientable surface with genus g. This is simply the connected sum of g tori. In this example we let $g = 2$ for simplicity. We can give Σ_g a simple CW complex structure. The CW complex chain is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Both boundary maps turn out to be the zero maps. We can even compute the generators so for example

$$H_1(\Sigma g) = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g.$$

From the universal coefficient theorem we get that

$$H^i(\Sigma_g;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0,2 \\ \mathbb{Z}^{2g}, & i = 1 \end{cases}.$$

Recall that

$$H^1(\Sigma_g;\mathbb{Z}) = \mathrm{Hom}(H_1(\Sigma_g),\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

and suppose it is generated by $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$. We also have that

$$H^2(\Sigma_g;\mathbb{Z}) = \mathrm{Hom}(H_2(\Sigma_g),\mathbb{Z})\mathbb{Z}.$$

so it is only generated by one element. We have that

$$\alpha_i(a_j) = \delta_{ij} \quad \text{and} \quad \alpha_i(b_j) = 0 \quad \text{and} \quad \beta_i(a_j) = 0 \quad \text{and} \quad \beta_i(b_j) = \delta_{ij}.$$

We are going to compute $\alpha_i \cup \beta_i$. Consider a simplicial structure for Σ_g (we are doing this for the formula from)

Let's be more practical and find α_1. We see that

$$\alpha_1(a_1) = 1 \quad \text{and} \quad \alpha_1(b_1) = \alpha_1(b_2) \quad \text{and} \quad \alpha_1(a_2) = 0.$$

Thus the cocycle α_1 is any map that connects a_1 and the other a_1 edges. Similarly for the rest of the cycles.

Accodring to the formula o the cup product we get that for T_1 (a triangle with edges e_1, e_2, a_1)

$$\alpha_1 \cup \beta_1(T_1) = \alpha_1(e_1) \cdots \beta_1(a_1) = 0$$

We also get

$$\alpha_1 \cup \beta_1(T_1) = \alpha_1(e_2) \cdots \beta_1(b_1) = 1$$

Thus $\alpha_1 \cup \beta_1(T_i) = 1$ if and only if $i = 2$ and otherwise it is equal to 0.

We proceed to look for the generator of $H_2(\Sigma_g)$. We see (by guesswork) that one element (cycle) with the image zero is

$$\sigma = T_1 + T_2 - T_3 - T_4 + T_5 + T_6 - T_7 - T_8.$$

Thus, since $H_2(\Sigma_g) \cong \mathbb{Z}$, and since we chose the minimal coefficients for $T_1, \ldots, T_8$ (± 1), so it is not a multiple of any other element (it can't be n times another element), so it is clear that σ generates $H_2(\Sigma_g)$. We want to find a generator for $H^2(H_2(\Sigma_g), \mathbb{Z})$. The generator is just the dual of σ which is the map

$$\sigma^*(\sigma) = 1$$

so $\sigma^* \in \operatorname{Hom}(H_2(\Sigma_2), \mathbb{Z})$ so σ^* is the generator. Therefore since $\alpha_1 \cup \beta_1(\sigma) = 1$ we get that $\alpha_1 \cup \beta_1 = \sigma^*$.

We proceed to compute $\beta_1 \cup \alpha_1$. We get that $\beta_1 \cup \alpha_1(T_i) = 1$ if and only if $i = 3$ and otherwise it is equal to 0. Thus $\beta_1 \cup \alpha_1(\sigma) = -1$ so $\beta_1 \cup \alpha_1 = -\sigma^*$. One may check that $\alpha_1 \cup \alpha_1 = \beta_1 \cup \beta_1 = 0$ and $\beta_2 \cup \alpha_2 = -\sigma^*$ and $\alpha_2 \cup \beta_2 = \sigma^*$. Recall that the cup product is bilinear so it makes computations easier.

Example 2.5. Let X be the connected sum of k copies of $\mathbb{RP}^2$. In this example we are working with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We can construct a CW complex for X as follows.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}^k$ is generated by $a_1, \ldots, a_k$. The attaching maps are sending $1 \in H_2(X; \mathbb{Z}/2\mathbb{Z})$ to $(2, 2, \ldots, 2) \in \mathbb{Z}^k$ and any $a \in H_1(X; \mathbb{Z}/2\mathbb{Z})$ to 0. Since we are working with coefficents in $\mathbb{Z}/2\mathbb{Z}$ both maps are zero maps (in cohomology?)

$$0 \longleftarrow \mathbb{Z}/2\mathbb{Z} \longleftarrow (\mathbb{Z}/2\mathbb{Z})^k \longleftarrow \mathbb{Z}/2\mathbb{Z} \longleftarrow 0$$

Denote generators of H^1 by $\alpha_1, \alpha_2, \ldots, \alpha_k$. We have that $\alpha_i(a_j) = \delta_{ij}$.

$$H^2(X; \mathbb{Z}/2\mathbb{Z}) \cong \operatorname{Hom}(H_2(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}).$$

$H_2(X; \mathbb{Z}/2\mathbb{Z})$ is generated by $\sigma = T_1 + T_2 + \cdots + T_{2k}$. $H^2(X; \mathbb{Z}/2\mathbb{Z})$ is generated by σ^*.

Notice that $\alpha_i \cup \alpha_j = 0$ when $i = j$. We see that $\alpha_i \cup \alpha_i(T_n) = 1$ if and only if $n = 2i$. Thus $\alpha_i \cup \alpha_i(\sigma) = 0 + \cdots + 1 + \cdots + 0 = 1$ so $\alpha_i \cup \alpha_i = \sigma^*$.

Theorem 2.6. *Let R be a commutative ring. Then $\alpha \cup \beta = (-1)^{k \cdot l} \beta \cup \alpha$ where $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$.*

Remark 2.23. In particular we have that when k and l are odd then $\alpha \cup \beta = -\beta \cup \alpha$. And that $\alpha \cup \alpha = -\alpha \cup \alpha$ so $2(\alpha \cup \alpha) = 0$.

Proof. We have that and since R is commutative

$$\beta \cup \alpha\left([v_n, \ldots, v_0]\right) = \alpha\left([v_n, \ldots, v_k]\right) \cdot \beta\left([v_n, \ldots, v_k]\right).$$

Let $\sigma\colon \{0,1\ldots,n\} \to \{0,1,\ldots,n\}$ be the permutation that sends $i \mapsto n-i$. Let $\rho\colon C_n \to C_n$ be given by $[v_0,\ldots,v_n] \mapsto \sigma[v_n,\ldots,v_0]$ We can verify that ρ is a chain map ($\partial\rho = \rho\partial$) that is homotopic to the identity (find $P\colon C_n \to C_{n+1}$ so that $\partial P + P\partial = \rho - \mathrm{id}$). The rest of the proof will not be added. Suppose ρ is chain homotopic to the identity. Let $\varphi \in H^k$ and $\psi \in H^l$,

$$\varphi \cup \psi \left([v_0,\ldots,v_{k+l}]\right) = \varphi\left([v_0,\ldots,v_k]\right) \cdot \psi\left([v_k,\ldots,v_{k+l}]\right).$$

We get that ρ^* so that $\psi^*\varphi = \varphi$ in cohomology. Then

$$\begin{aligned}\left(\rho^*\varphi \cup \rho^*\psi\right)[v_0,\ldots,v_{k+1}] &= \rho^*\varphi[v_0,\ldots,v_k]\rho^*\psi[v_k,\ldots,v_{k+l}]\\ &= \varphi\rho[v_0,\ldots,v_k]\psi\rho[v_k,\ldots,v_{k+l}]\\ &=\end{aligned}$$

to be completed? □

Lemma 2.7. *Let $f\colon X \to Y$ be continuous, and let R be a commutative ring. Then for $f^*\colon H^n(Y;R) \to H^n(X;R)$*

$$f^*\varphi \smile f^*\psi = f^*(\varphi \smile \psi).$$

In fact,

$$f^{\#}\varphi \smile f^{\#}\psi = f^{\#}(\varphi \smile \psi).$$

2.3 The cohomology ring

Remark 2.24. Let $H^*(X;R) = \bigoplus_{k\geq 0} H^k(X;R)$ be equipped with the cup product

$$\smile\colon H^*(X;R) \times H^*(X;R) \to H^*(X;R)$$

defined on $\varphi \in H^k$ and $\psi \in H^l$ as we saw before, and extended to $H^*(X;R) \times H^*(X;R)$ blinearly.

Remark 2.25. Recall that $\smile$ be bilinear (distributive) on $H^k \times H^l \to H^{k+l}$.

Example 2.6. Let $\varphi \in H^{k_1}$, $\varphi \in H^{k_2}$, $\psi_1 \in H^{l_1}$ and $\psi_2 \in H^{l_2}$. Then

$$(\varphi_1 + \varphi_2) \smile (\psi_1 + \psi_2) =$$

Remark 2.26. If R has a unit $1 \in R$ then $H^*(X;R)$ has an identity element $1 \in H^0(X;R)$ so that $1\left([v]\right) = 1 \in R$.

Definition 2.5 (ℕ-graded ring)**.** A ℕ-graded ring is a ring of the form $A = \bigoplus_{k\in\mathbb{N}} A_k$ such that $A_k \cdot A_l \subseteq A_{k+l}$.

Remark 2.27. If $a \in A_k$ then we denote $|a| = k$.

Remark 2.28. We say that a graded ring A is graded commutative if for $a \in A_k$ and $b \in A_l$ we have that $ab = (-1)^{kl}ba$ or $ab = (-1)^{|a|\cdot|b|}ba$.

Example 2.7. The ring of polynomials $P = \bigoplus_{k\in\mathbb{N}} \mathrm{Sp}\{x^k\}$ is an ℕ-graded ring.

Example 2.8 (Exterior algebra)**.** We define $\Lambda_R[x_1,\ldots,x_n]$ the be the polynomials in $x_1,\ldots,x_n$, but we require $|x_i| = 1$ (the dimension of x_i is 1) which implies that $x_ix_j = x_jx_i$ and we require $x_ix_i = 0$ (this is also implied from the requirement if $2 \neq 0$). We have that

$$\begin{aligned}\Lambda_R[x_1,\ldots,x_n] &= (R\cdot 1)\,(A_0)\\ &= (Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n)\,(A_1)\\ &\quad\oplus (Rx_1x_2 \oplus Rx_1x_3 \oplus \cdots \oplus Rx_{n-1}x_n)\\ &\quad\vdots\\ \oplus\,(Rx_1x_2\cdots x_n)\end{aligned}$$

Theorem 2.8. *The cohomology ring of $\mathbb{RP}^n$ (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is $\mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1}) = \mathbb{Z}/2\mathbb{Z}\cdot 1 \oplus \mathbb{Z}/2\mathbb{Z}x \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}x^n$ where $H^k = \mathbb{Z}/2\mathbb{Z}x^n$ is of grade k.*

Proof. □

Remark 2.29. We that $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$ and $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$.

Remark 2.30. Although we have that the reduced homology and cohomology of $S^2 \vee S^4$ and $\mathbb{CP}^2$ are equal, their cohomology rings are not equal $H^*(S^2 \vee S^4) \neq H^*(\mathbb{CP}^2)$. That is because

$$H^*(S^2 \vee S^4) = H^*(S^2) \oplus H^*(S^4) \neq \mathbb{Z}[x]/(x^3) = H^*(\mathbb{CP}^2).$$

2.4 Künneth formulas

Remark 2.31. The goal of this subsection is to describe the cohomology ring $H^*(X \times Y)$ in terms of $H^*(X)$ and $H^*(Y)$.

Definition 2.6 (Cross product)**.** The cross product is a map $H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y)$. Let $\varphi \in H^i(X)$ and $\psi H^j(Y)$, and let $\rho_X \colon X \times Y \to X$ and $\rho_Y \colon X \times Y \to Y$ be the projections on X and Y respectively. Then the cross product is the map

$$(\varphi, \psi) \mapsto (\rho_X^*\varphi) \smile (\rho_Y^*\psi) := \varphi \times \psi.$$

Note that $\rho_X^*\varphi \in H^i(X \times Y)$ and $\rho_Y^*\psi \in H^j(X \times Y)$.

Definition 2.7 (Tensor product of spaces)**.** Let U and V be vector spaces. Then $U \otimes V$ is the unique vector space such that the diagram

$$\begin{tikzcd} U \times V \arrow[rr, "f"] \arrow[dr, "\otimes"'] & & W \\ & U \otimes V \arrow[ur, "\exists ! g"'] & \end{tikzcd}$$

commutes for any vector space W and any bilinear map $f \colon U \times V \to W$.

Remark 2.32. We have that

$$U \otimes V = \operatorname{span}_{\mathbb{F}}(U \times V)/(u_1 + u_2, v) = (u_1, v) + (u_2, v) \cdots$$

Remark 2.33. Let B_U be a basis for U and let B_V be a basis for V. Then $U \otimes V$ has basis $\{\alpha \otimes \beta \mid \alpha \in B_U \text{ and } \beta \in B_V\}$. In particular $\dim(U \otimes V) = \dim(U)\dim(V)$.

Theorem 2.9. *Let X and Y be CW complexes. Then there exists an isomorphism*

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

given by the cross product $\alpha \otimes \beta \mapsto \alpha \times \beta$. Recall that $H^*(X) = \bigoplus H^i(X)$ etc.

Proposition 2.10. *For any $\alpha \in H^*(X)$ and $\beta \in H^*(Y)$ we have*

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') = (-1)^{|\alpha'||\beta|}(\alpha\alpha') \otimes (\beta\beta')$$

Corollary 2.11. *We have that $H^i(T^n;\mathbb{F}) = \mathbb{F}^{\binom{n}{i}}$.*

Remark 2.34. We are not going to prove this corollary, we are going to prove a more general corollary.

Corollary 2.12. *We have that*

$$H^i(X \times Y) = \bigoplus_{k=0}^{i} H^k(X) \otimes H^{i-k}(Y) \cong H^i(X \times Y).$$

In particular

$$\sum_{k=0}^{i} \dim(H^k(X)) \dim(H^{i-k}(Y)) = \dim(H^i(X \times Y)).$$

3 Manifolds and Poincaré duality

3.1 Introduction

Definition 3.1 (Topological n-manifold)**.** Let M be a second countable Hausdorff space. Then M is said to be an n-manifold if M is locally homeormophic to $\mathbb{R}^n$. Moreover, M is a closed n-manifold if M is compact.

Remark 3.1. We say that M is closed if it is compact.

Example 3.1.

Theorem 3.1 (Poincaré duality)**.** *Let M be a closed connected n-manifold. Then*

$$H_i(M; \mathbb{Z}/2\mathbb{Z}) \cong H_{n-i}(M; \mathbb{Z}/2\mathbb{Z}).$$

In particular $H_n(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

3.2 Orientation

Remark 3.2. Let M be an n-manifold. Recall that the local homology of M at $x \in M$ is given by $H_i(M \mid x) := H_i(M, M - \{x\})$. More generally for A we denote $H_i(M \mid A) = H_i(M, M - A)$.

By excision by any neighbourhood U of x we have

$$H_i(M \mid x) \cong H_i(U \mid x) \cong H_i(\mathbb{R}^n \mid x) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n. \end{cases}$$

For any $x, y \in B \subseteq U \cong \mathbb{R}^n$, since $U - \{x\} \simeq_{\text{dfr}} U - B$, we have

$$\begin{aligned} H_n(M \mid x) &= H_n(U \mid x) \\ &\cong H_n(U, U - \{x\}) \\ &\cong H_n(U, U - B) \\ &\cdots \\ &\cong H_n(M \mid y) \end{aligned}$$

Definition 3.2 ($\mathbb{Z}$-orientation)**.** A $\mathbb{Z}$-orientation on M is a choice $x \mapsto \mu_x \in H_n(M \mid x; \mathbb{Z}) \cong \mathbb{Z}$ such that

(1) μ_x generates $H_n(M \mid x)$.

(2) μ_x is consistant. That is, if $x, y \in U \cong \mathbb{R}^n$ for some $U \subseteq M$ then $H_n(M \mid x) \cong H_n(U, U - B) \cong H_n(M \mid y)$ sends μ_x to μ_y.

Definition 3.3 (Orientable manifold)**.** We say that M is orientable if it admits an orientation. In fact, M is orientable if and only if it admits exactly two orientations.

Example 3.2. The open Möbius band is non-orientable.

Definition 3.4 ($\mathbb{Z}/2\mathbb{Z}$-orientation)**.** A $\mathbb{Z}/2\mathbb{Z}$-orientation on M is a choice $x \mapsto \mu_x \in H_n(M \mid x; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ such that

(1) μ_x generates $H_n(M \mid x)$.

(2) μ_x is consistant. That is, if $x, y \in U \cong \mathbb{R}^n$ for some $U \subseteq M$ then $H_n(M \mid x) \cong H_n(U, U - B) \cong H_n(M \mid y)$ sends μ_x to μ_y.

Remark 3.3. Notice that every surface M is $\mathbb{Z}/2\mathbb{Z}$ orientable. That is because every $x \in M$ is send to 1, and this satisfies our conditions for orientation.

Definition 3.5 ($\mathbb{Z}$-section)**.** A $\mathbb{Z}/2\mathbb{Z}$-section on M is a choice $x \mapsto \mu_x \in H_n(M \mid x; \mathbb{Z}) \cong \mathbb{Z}$ such that

(1) μ_x is consistant. That is, if $x, y \in U \cong \mathbb{R}^n$ for some $U \subseteq M$ then $H_n(M \mid x) \cong H_n(U, U - B) \cong H_n(M \mid y)$ sends μ_x to μ_y.

Remark 3.4. Let M be a manifold with a $\mathbb{Z}$-section μ. Then we define

$$\widetilde{M} := \left\{(x, \mu_x) \mid x \in M \text{ and } \mu_X \text{ generates } H_n(M \mid x)\right\}.$$

Theorem 3.2. *Let M be a closed connected n-manifold. Then*

(1) If M is an R-orientable, then the the map $\varphi\colon H_n(M) \to H_n(M \mid x; R)$ is an isomorphism.

(2) If M is not $\mathbb{Z}$-orientable, then $H_n(M; \mathbb{Z}) = 0$.

(3) $H_i(M) = 0$ for $i > n$.

Remark 3.5. A closed connected n-manifold M is orientable if and only if $H_n(M : \mathbb{Z}) = \mathbb{Z}$. More generally, a closed connected n-manifold M is orientable if and only if $H_n(M : \mathbb{Z}) \neq 0$.

Lemma 3.3. *Let M be an n-manifold and let $A \subseteq M$ be compact.*

(i) If $x \mapsto \alpha_x$ is a section, then there exists a unique class $\alpha_A \in H_n(M \mid A)$ such that for all $x \in A$ we have that $\alpha_A \mapsto \alpha_x$ in $H_n(M \mid A) \to H_n(M \mid x)$.

(ii) $H_i(M \mid A) = 0$ for all $i > n$.

Proof of Theorem 3.2. Here we use the lemma to prove Theorem 3.2. □

Proof of Theorem 3.3. to be added □

Theorem 3.4. *Let M be a noncompact connected n-manifold. Then $H_i(M) = 0$ for all $i \geq n$.*

Proof. □

Remark 3.6. Last lecture we have seen that if M is an orientable manifold then

$$H_i(M; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i > n, \\ ? & \text{otherwise.} \end{cases}$$

Lemma 3.5. *Let $x \mapsto \alpha_x \in H_n(M \mid x)$ be a section and let $A \subseteq M$ be compact. Then there exists a unique $\alpha_A \in H_n(M \mid A)$ so that for all $x \in A$ we have $\alpha_x \in H_n(M \mid x)$. Moreover, $H_i(M \mid A) = 0$ for all $i > n$.*

Theorem 3.6. *Let M be a noncompact n-manifold. Then $H_i(M; \mathbb{Z}) = 0$ for all $i \geq 0$.*

Proof. Let $z \in H_i(M; \mathbb{Z})$ and let $U \subseteq M$ be an open subset such that U contains the simplicies of z such that $\overline{U}$ is compact. We have $V := M - \overline{U}$.

Consider the long exact sequence for the triple $(M, U \cup V, V)$.

$$\cdots \longrightarrow H_n(U \cup V, V) \longrightarrow H_n(M, V) \longrightarrow H_n(M, U \cup V) \longrightarrow H_{n-1}(U \cup V, V) \longrightarrow \cdots$$

proof not to be added. using this sequence and the lemma. □

3.3 Kervaire–Laudinbach conjecture

Let $1 \neq G = \langle S \mid R \rangle$. Then for all $r \in F(s,t)$ for some new element t we have

$$G * \langle r \rangle / \ll r \gg = \langle s, t \mid R, r \rangle \neq 1.$$

For example in a general relation is of the form

$$r = g_0 t^{\epsilon_0} g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} g_n$$

for $\epsilon_i \in \mathbb{Z}$ and $g_i \in \mathbb{Z}$.

We clearly have that if $\langle G, t \mid r, G \rangle$ is nontrivial then $\langle G, t \mid r \rangle$ is clearly nontrivial. We see that

$$\langle G, t \mid r, G \rangle = \langle t \mid t^{\exp_t(r)} \rangle$$

where we define $\exp_t(r) := \sum \epsilon_i$. $\langle t \mid t^{\exp_t(r)} \rangle$ is nontrivial when $\exp_t(r) \neq \pm 1$ so the only intresting cases are when $\exp_t(r) = \pm 1$. Without loss of generality we can see assume that $\exp_t(r) = 1$.

A stronger version of the conjecture is that when $\exp_t(r) \neq 0$ or G is torsion free then there exists an embedding $G \hookrightarrow \langle G, t \mid r \rangle$.

For example consider S_3.. (it is finite and exp is zero)

Theorem 3.7 (Gerstenhaber–Rothams)**.** *The Kervaire–Laudinbach conjecture holds for finite G.*

Proof. We will show that when G is finite that there exists an embedding $G \hookrightarrow \langle G, t \mid r \rangle$.

In order to prove this it suffices to find $G \leqslant \widetilde{G}$ and $\bar{g} \in \widetilde{G}$ such that $r(\tilde{g}) = 1$.

By Caylay's theorem $G \leqslant S_n$ and $S_n \curvearrowright \mathbb{C}^n$ by $\sigma e_i = e_{\sigma(i)}$. This is a map $S_n \to U(n) := \widetilde{G}$ where $U(n) := \{A \in M_{n\times n}(\mathbb{C}) \mid A^* A = I\}$.

This is a group, which is also a connected and closed manifold. Therefore $H_n(\widetilde{G}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. (why) Actually we even have $H_n(\widetilde{G}; \mathbb{Z}) = \mathbb{Z}$ (dont need to know why).

Since M is a connected comapact orientable n-manifold, we can define a degree map $f \colon M \to M$ with $\deg(f) = k$ if $f_* \colon H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{Z})$ (which is a map $f_* \colon \mathbb{Z} \to \mathbb{Z}$) maps $m \mapsto m \cdot k$. We need to find $\tilde{g} \in \widetilde{G}$ so that for $r \colon \widetilde{G} \to \widetilde{G}$ defined by $r(x) = g_0 x^{\epsilon_1} g_1 \cdots x^{\epsilon_n} g_n$ we have $r(\tilde{g}) = 1$. If $\forall \tilde{g} \in \widetilde{G}$ we have that $r(\tilde{g}) \neq 1$ then since

$$\begin{tikzcd}
\widetilde{G} \arrow[rr] \arrow[dr] & & \widetilde{G} \\
& \widetilde{G} - \{1\} \arrow[ur] &
\end{tikzcd}$$

commmutes and $H_n(\widetilde{G} - \{1\}) = 0$ we get that $\deg(r) = 0$.

On the other hand we can choose paths γ_i that maps continuously each "coefficient" g_i of $r(x) = g_0 x^{\epsilon_1} g_1 \cdots x^{\epsilon_n} g_n$ to 1. Therefore there exists a homotopy between $r(x)$ to

$$g_0 x^{\epsilon_1} g_1 \cdots x^{\epsilon_n} g_n = x^{\exp_t(r)} = x.$$

This means that $r(x)$ is homotopic to the identity and so

$$\deg(r) = \deg(\mathrm{id}) = 1.$$

This contradiction completes the proof. $\square$

3.4 Poincaré duality

Remark 3.7. If M is triangulated (M is homeomorphic to a (PL)(partially linear) simplicial complex X).

Theorem 3.8 (Poincaré duality)**.** $H^k(M;\mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z}/2\mathbb{Z})$.

Proof. nah uh □

Definition 3.6 (Cap product)**.** Let $l \leq k$. Then the cap product is a map $C_k \times C^l \xrightarrow{\frown} C_{k-l}$. It is defined by

$$\sigma \frown \varphi = \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \cdot \sigma|_{[v_l,\dots,v_k]}$$

where $\sigma \colon \Delta^k = [v_0,\dots,v_k] \to X$.

Remark 3.8. We have that $\psi(\sigma \frown \varphi) = (\psi \smile \varphi)(\sigma)$ where $\varphi \in C^{k-l}$ and $\sigma \in C_k$ and $\varphi \in C^l$ and $(\psi \smile \varphi) \in C^k$.

Remark 3.9. The cap product is also well defined as a map $\frown H_n(X) \times H^k(X) \to H_{n-k}(X)$.

Lemma 3.9. $\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$.

Proof. Since $\frown$ is bilinear we get that

$$\partial[v_0,\dots,v_n] \frown \varphi = \sum_{i=0}^{k} (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n] \frown \varphi.$$

Then

$$\begin{aligned}\partial[v_0,\dots,v_n] \frown \varphi &= \sum_{i=0}^{k} (-1)^i [v_0,\dots,\cap v_i,\dots,v_n] \frown \varphi.\\ &= \sum_{i=0}^{l} (-1)^i \varphi\left([v_0,\dots,\cap v_i,\dots,v_{l+1}]\right) \cdot [v_{l+1},\dots,v_k]\\ &+ \sum_{i=l+1}^{k} (-1)^i \varphi\left([v_0,\dots,v_l]\right) \cdot [v_{l+1},\dots,\hat{v}_i,\dots,v_k].\end{aligned}$$

We have that

$$(*) \quad \partial(\sigma \frown \varphi) = \sum_{i=l+1}^{k} (-1)^{i-l} \varphi\left([v_0,\dots,v_l]\right) \cdot [v_{l+1},\dots,\hat{v}_i,\dots,v_k].$$

Also, we know that

$$\begin{aligned}(\#) \quad \sigma \frown \delta\varphi &= (\varphi\partial)\left([v_0,\dots,v_{l+1}]\right) \cdot [v_{l+1},\dots,v_k]\\ &= \sum_{i=0}^{l+1} (-1)^i \varphi\left([v_0,\dots,\hat{v}_i,\dots,v_{l+1}]\right) \cdot [v_{l+1},\dots,v_k]\end{aligned}$$

We also have that

$$\left(l\text{-term of } (*)\right) = (-1)^{l+1}\left((l+1)\text{-term of } (\#)\right).$$

From this we can derive the proof □

Remark 3.10. The cap product, we defined on relative homology is a map $\frown \colon H_n(X,A) \times H^k(X,A) \to H_{n-k}(X)$.

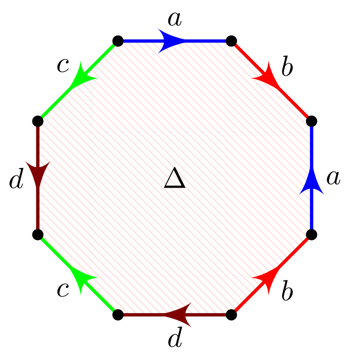
Remark 3.11. The cap product is functorial. That is, $f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$.

Definition 3.7 (Fundamental class)**.** Let M be a closed oriented topological manifold. The class $\mu := [M] \in H_n(M) \mapsto \mu_x \in H_n(M \mid x)$ is called the fundamental class of M.

Theorem 3.10 (Poincaré duality)**.** *Let M be a closed oriented topological manifold. Then the map $D_M\colon H^i(M) \to H_{n-i}(M)$ defined by $D_M = [M] \frown$ is an isomorphism.*

Remark 3.12. Poincaré duality also holds without the assumption that M is orientable so long as we consider homology and cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Example 3.3. Consider $X = \Sigma_2$.



The fundamental class $[M] \in H_1(X)$ is

$$a + b - a - b + d + c - d - c$$

We know that $H^1(X) = \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\gamma + \mathbb{Z}\delta$. We see that $\alpha_1(a_1) = 1$ When we compute $D_M(\alpha_1)$ we get

$$D_M(\alpha_1) = [M] \frown \alpha_1 = -\tau_1 \frown \alpha_1 = -\alpha_1(e_1)a_1 - \alpha_1(e_2)b_1 + \alpha_1(e_1)a_1 = -b_1.$$

Definition 3.8 (Compactly supported cohomology)**.** Let X be a locally finite (Δ, CW, or simplicial) complex. Then

$$C_C^k := \left\{ f \in C^k \mid f \text{ is supported on finitely many simplices} \right\}.$$

Check that $\delta\colon C_C^k \to C_C^{k+1}$.

Remark 3.13. When $K \subset X$ is compact we denote $K \subset\subset X$.

Remark 3.14. In singular cohomology we get that

$$C_C^k(X) = \bigcup_{K \subset\subset X} C^k(X, X - K).$$

or

$$C_C^k(X) = \left\{ f\colon \text{singular } k\text{-simplices} \to \mathbb{Z} \mid \exists K \text{ s.t. } f(\sigma) := f(X - K) = 0 \right\}.$$

Remark 3.15. A partially order set $(X, \leq)$ is said to be directed if for all $i, j \in I$ there exists $k \in I$ so that $k \geq i, j$.

Definition 3.9 (Direct limit)**.** Let $\{A_i\}_{i \in I}$ be a collection is of abelian groups so that $(I, \leq)$ is a directed partially ordered sets. For all $i \leq j$ let $f_{ij}\colon A_i \to A_j$ be maps. Then

$$\begin{aligned} \varinjlim A_i &= \bigoplus A_i / \langle \{ a_i - f_{ij}(a_i) \mid i \leq j \text{ and } a_i \in A_i \} \rangle \\ &= \bigsqcup A_i / a_i \sim f_{ij} a_i \quad \forall i \leq j \text{ and } a_i \in A_i. \end{aligned}$$

Example 3.4. The sets $(\mathbb{N}, \leq)$ and $(K, \subseteq)$ where K is the set of all compact subsets of a topological space X.

Remark 3.16. We have that $H_C^K(X) = \varinjlim H^k(X, X - K)$ for $K \subset\subset X$.

Definition 3.10 (Cofinal set)**.** We say that $J \subseteq I$ is cofinl if for all $i \in I$ there exists $j \in J$ such that $i \leq j$.

Definition 3.11 (Proper map)**.** A map so that the preimage of a compact set is compact is called proper.

Remark 3.17. We have that $H_i(X) = \varinjlim_{K \subseteq X} H_i(K)$.

Remark 3.18. We have that $D_M\colon \varinjlim_K H^i(M \mid K) \to H_{n-i}(M)$.

Theorem 3.11 (Poincaré duality)**.** *Let M be an oriented n-manifold with orientation $x \mapsto \mu_x$. Then D_M is an isomorphism.*