

Algebraic Geometry

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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1 Introduction

1.1 Algebraic subsets of $\mathbb{C}^n$ and Hilbert's nullstellensatz

Algebraic geometry deals with sets of solutions to systems of polynomial equations.

Example 1.1. The solution set for $xy - 1 = 0$ in $\mathbb{C}^2$.

Example 1.2. The solution set for $x^n + y^n = 1$ in $\mathbb{Q}^2$ for $n \geq 3$. By Fermat's last theorem there are no solutions to this equation unless either x or y is zero.

Example 1.3. The solution set for the following system of polynomials:

$$\begin{cases} xt - yz = 0 \\ xt - y^2 = 0 \\ yt - z^2 = 0 \end{cases}$$

in $\mathbb{C}^4$.

Example 1.4. Consider an $n \times n$ matrix A as a point in $\mathbb{C}^{n^2}$. Then we have that

$$\{A \mid \text{tr}(A) = 0\} = \left\{ A \mid \sum_{i=1}^n a_{ii} = 0 \right\}$$

is given as the solution set of a polynomial. Similarly

$$\{\det A = 0\} = \left\{ \sum_{\tau \in S_n} \text{sgn}(\tau) a_{1,\tau(1)} a_{2,\tau(2)} \cdots a_{n,\tau(n)} = 0 \right\},$$

is given by the solution set for a polynomial in the entries of the matrix (by the formula with).

Definition 1.1 (Set of zeros). For a subset $S \subseteq \mathbb{C}[x_1, \dots, x_n]$, we define its set of zeros by

$$Z(S) := \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid f(a_1, \dots, a_n) = 0 \quad \forall f \in S\}.$$

Proposition 1.1. Here are some basic properties of sets of zeros:

- (1) $Z(S) = Z((S))$ where (S) is the ideal generated by S . Thus $Z(S) = Z(\{f_1, \dots, f_m\})$ for any set $\{f_1, \dots, f_m\}$ such that $(f_1, \dots, f_m) = (S)$. Such a finite generating set exists by the Hilbert basis theorem, which tells us that $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian.
- (2) If $I \subseteq J$ then $Z(I) \supseteq Z(J)$.
- (3) $Z(I + J) = Z(I) \cap Z(J)$,
- (4) $Z(I \cap J) = Z(IJ) = Z(I) \cap Z(J)$.
- (5) $Z(\sqrt{I}) = Z(I)$, we so can restrict to radical ideals.

Remark 1.1. Recall that the radical of an ideal $I \triangleleft R$ is defined by

$$\sqrt{I} := \left\{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f^k \in I \text{ for some } k \geq 1 \right\}.$$

Example 1.5. We have that $Z(\{0\}) = Z((0)) = \mathbb{C}^n$, and $Z(\mathbb{C}[x_1, \dots, x_n]) = Z(\{1\}) = \emptyset$. Additionally, we have that $Z((x_1 - a_1, \dots, x_n - a_n)) = \{(a_1, \dots, a_n)\}$. Observe that $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal.

Definition 1.2. For a subset $V \subseteq \mathbb{C}^n$ we define the ideal:

$$I(V) := \{f \in \mathbb{C}[x_1, \dots, x_n] \text{ such that } f(p) = 0 \text{ for all } p \in V\}.$$

Equivalently,

$$I(V) := \bigcap_{p \in V} m_p,$$

where $m_p = (x_1 - a_1, \dots, x_n - a_n)$ is the maximal ideal corresponding to the point $p = (a_1, \dots, a_n)$.

Definition 1.3 (Zariski topology). The Zariski topology on $\mathbb{C}^n$ is defined by declaring the closed sets to be $Z(I)$.

Remark 1.2. Note that the Zariski topology is very weak, in the sense that it does not have a lot of open or closed sets. In particular, it is not Hausdorff.

Question. For $n = 1$ what are proper closed subsets of $\mathbb{C}$?

Answer. If $0 \neq I \subsetneq \mathbb{C}[x]$, then since $\mathbb{C}[x]$ is a PID and algebraically closed $I = (f(x))$ for some polynomial $f(x) = \prod (x - a_i)$. Thus the Zariski closed subsets of $\mathbb{C}$ are precisely the finite sets. This implies that in this case, the Zariski topology agrees with the cofinite topology.

Theorem 1.2 (Weak Hilbert's nullstellensatz). *All maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$ for some $(a_1, \dots, a_n) \in \mathbb{C}^n$.*

Corollary 1.3. *Let $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$. Then $Z(I) = \emptyset$ if and only if $I = (1) = \mathbb{C}[x_1, \dots, x_n]$.*

Proof. We have seen that if $I = (1)$ then $Z(I) = \emptyset$. If $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$ then by applying Zorn's lemma $I \subseteq m_p$ for some maximal ideal m_p , which is a point ideal for some point $p = (a_1, \dots, a_n)$ by the weak nullstellensatz theorem, thus $p \in Z(I)$. This implies that $Z(I) \neq \emptyset$, which completes the proof. $\square$

Remark 1.3. As a consequence of the weak nullstellensatz theorem, the set $Z(I)$ can be identified with the set of maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$ which contain I , which in turn are in natural bijection to the set of maximal ideals of the quotient ring of functions $k[X] := \mathbb{C}[x_1, \dots, x_n]/I$ from the correspondence theorem.

Theorem 1.4 (Regular strength Hilbert's nullstellensatz). *For any ideal I in $\mathbb{C}[x_1, \dots, x_n]$ we have*

$$I(Z(I)) = \sqrt{I}.$$

That is, $I(Z(I))$ is the closure of I with respect to the property of being radical.

Proof. If $f \in \sqrt{I}$, then $f^k \in I$ for some $k \geq 1$. Therefore $f^k(p) = (f(p))^k = 0$ for all $p \in Z(I)$. This implies that $f(p) = 0$ for all $p \in Z(I)$. which implies that $f \in I(Z(I))$.

Conversely (we are using Rabinowitch's trick, which is a particular case of localization). Consider $0 \neq f \in I(Z(I))$. In $\mathbb{C}[x_1, \dots, x_n, t]$ we claim that the ideal

$$\hat{I} = (ft - 1) + (I) \subseteq \mathbb{C}[x_1, \dots, x_n, t]$$

has no zeros, where (I) denotes the ideal in the ring $\mathbb{C}[x_1, \dots, x_n, t]$ generated by I . Indeed, for any $(a_1, \dots, a_n, b) \in Z(\hat{I})$ there must exist $g \in I$ such that $g(a_1, \dots, a_n) = 0$, and in particular $f(a_1, \dots, a_n) = 0$, so we get that $(ft - 1)$ has to vanish at $(a_1, \dots, a_n, b)$, or in other words that $0 = f(a_1, \dots, a_n)b - 1 = -1$ which is a contradiction to $f(a_1, \dots, a_n) = 0$.

By Theorem 1.3, the ideal $\hat{I}$ must be the whole ring, and in particular it contains $1 \in \hat{I}$. This means that there exist $f_i \in I$ such that:

$$1 = (f(x_1, \dots, x_n)t - 1)g_0(x_1, \dots, x_n, t) + \sum_i f_i(x_1, \dots, x_n)g_i(x_1, \dots, x_n, t).$$

Consider the image of this equality in the field of rational functions in n variables $\mathbb{C}(x_1, \dots, x_n)$ under the algebra map which sends x_i to x_i and t to $f(x_1, \dots, x_n)^{-1}$. Then the equation becomes:

$$1 = 0 + \sum_i f_i(x_1, \dots, x_n) \frac{h_i(x_1, \dots, x_n)}{f_i(x_1, \dots, x_n)^{k_i}},$$

for some polynomials h_i and some $k_i \geq 0$. After clearing the denominators we conclude that f^k lies in I , which proves the other side of the inclusion and concludes the proof. $\square$

Remark 1.4. Theorem 1.4 says that there is a one-to-one correspondence between closed algebraic subsets of $\mathbb{C}^n$ (i.e. the $Z(I)$) and radical ideals of $\mathbb{C}[x_1, \dots, x_n]$.

1.2 Irreducible components, dimension theory. Projective spaces. Algebraic varieties

Definition 1.4 (Reducible zero locus). $Z(I)$ is reducible if it can be written as a union of two distinct proper closed subsets $Z(I) = Z(I_1) \cup Z(I_2)$. It is called irreducible otherwise.

Example 1.6. For example when $n = 2$ and $I = (x_1, x_2)$ we can write $Z(I) = Z(x_1) \cup Z(x_2)$.

Remark 1.5. Disconnected algebraic sets are always reducible, however some reducible algebraic sets are not disconnected. You may try to think of some examples by yourselves, but some nice examples are also written in the corresponding part in Lev's notes.

Theorem 1.5. Every closed subset $Z(I)$ can be uniquely written as a union of irreducible closed subsets of $Z(I)$ none of which is contained in another.

Proof. First we will prove existence. Assume, by way of contradiction that the claim is false for some $Z(I)$. Then we can keep decomposing it into smaller unions of smaller and smaller subsets indefinitely. This indefinitely decreasing sequence of closed algebraic subsets correspond to an indefinitely increasing sequence of ideals in $\mathbb{C}[x_1, \dots, x_n]$ in contradiction to it being Noetherian. This implies we can write $Z(I)$ as a finite union of irreducible subsets, and it remains to remove subsets that are contained in others.

In the uniqueness direction, suppose we have two finite decompositions as described in the theorem:

$$Z(I) = \bigcup_{\alpha} Z(I_{\alpha}) = \bigcup_{\beta} Z(I_{\beta}).$$

For each α we have that

$$Z(I_{\alpha}) = \bigcup_{\beta} (Z(I_{\beta}) \cap Z(I_{\alpha})).$$

By irreducibility of $Z(I_{\alpha})$ one of $Z(I_{\beta}) \cap Z(I_{\alpha})$ must equal $Z(I_{\alpha})$. This implies that for every α there exists β such that $Z(I_{\alpha}) \subseteq Z(I_{\beta})$. Of course, the same is true for each β , so we get that $Z(I_{\alpha}) \subseteq Z(I_{\beta}) \subseteq Z(I_{\alpha'})$, and thus $Z(I_{\alpha}) \subseteq Z(I_{\alpha'})$, which implies that $\alpha = \alpha'$ and thus $Z(I_{\alpha}) = Z(I_{\beta})$. Since this works for each α and β we have proved uniqueness. $\square$

Remark 1.6. The decomposition of $Z(I)$ into irreducible components is a particular example of the commutative algebra statement that for any ideal I in a Noetherian ring, the radical $\sqrt{I}$ can be uniquely written as:

$$\sqrt{I} = \bigcap_i p_i,$$

where p_i are a finite set of minimal primes that contain I .

Definition 1.5 (Algebraic subvariety). If $Z(I)$ is irreducible, $Z(I)$ is called an algebraic subvariety of $\mathbb{C}^n$.

Remark 1.7. We note that algebraic subvarieties $Z(I)$ are very similar (and indeed not very far from) smooth complex manifolds. We have a concept of dimension for complex manifolds, so it is natural to expect a notion of dimension on $Z(I)$ as well.

Definition 1.6 (Dimension of subvarieties). For an irreducible $Z(I)$ the ideal I is prime (by exercise), so we can look at the quotient ring $A = \mathbb{C}[x_1, \dots, x_n]/I$ which is an integral domain. Its fraction field $\text{Frac}(A)$ is a finitely generated field extension of $\mathbb{C}$, and we define:

$$\dim_{\mathbb{C}} Z(I) := \text{tr. deg.}_{\mathbb{C}} \text{Frac}(A),$$

where $\text{tr. deg.}_{\mathbb{C}} \text{Frac}(A)$ represents the transcendence degree of $\text{Frac}(A)/\mathbb{C}$.

Definition 1.7 (Krull dimension). For a topological space X the Krull dimension of X is $\sup n$ where $Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ is an increasing chain of irreducible closed subsets of X . Similarly, the Krull dimension of a ring R is the supremum over the length of all increasing chains of prime ideals.

Remark 1.8. Note that $\text{Frac}(\mathbb{C}[x_1, \dots, x_n]/I)$ is simply the fraction field of the coordinate ring $k[x_1, \dots, x_n]$, which are the functions over X (we have seen this in commutative algebra). Similarly, we intuitively look at $\text{Frac}(\mathbb{C}[x_1, \dots, x_n]/I)$ as the field of “rational functions on $X = Z(I)$ ”.

Example 1.7. Clearly, the dimension of $\mathbb{C}^n$ is the transcendence degree of the field of rational functions $\mathbb{C}(x_1, \dots, x_n)$ which is n .

Remark 1.9. Any chain of irreducible closed subsets in $\mathbb{C}^n$ can be completed to a chain with a member in each dimension. The corresponding property for chains of prime ideals in $\mathbb{C}[x_1, \dots, x_n]$ is called *catenary*. Rings with this property are called catenary rings.

Remark 1.10. Algebraic geometers have a very annoying (for beginners) of drawing complex varieties of dimension n as objects of real dimension n , with the exception of Riemann surfaces (which are of complex dimension 1), and are sometimes drawn as real surfaces.

Definition 1.8 (Complex projective space). The complex projective space denoted $\mathbb{CP}^n$ is defined as the set

$$(\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

where $\mathbf{a} = (a_0, \dots, a_n) \sim \mathbf{b} = (b_0, \dots, b_n)$ if and only if there exists $\lambda \in \mathbb{C}^\times$ such that $\mathbf{a} = \lambda \mathbf{b}$.

Remark 1.11. The points in $\mathbb{CP}^n$ are written $(a_0 : a_1 : \dots : a_n)$ in order to distinguish them from points in $\mathbb{C}^{n+1}$.

Remark 1.12. We can think of $\mathbb{CP}^n$ as all the lines passing through the origin of $\mathbb{C}^{n+1}$.

Remark 1.13. For every $i \in \{0, 1, \dots, n\}$ we define the subspace

$$U_i = \{(x_0 : \dots : x_i : \dots : x_n) \in \mathbb{CP}^n \mid x_i \neq 0\}.$$

By the structure of $\mathbb{CP}^n$, these points can be normalized so that:

$$U_i = \left\{ \left(\frac{x_0}{x_i} : \dots : 1 : \dots : \frac{x_n}{x_i} \right) \in \mathbb{CP}^n \right\} \cong \mathbb{C}^n.$$

One can use the covering $\{U_i\}$ of $\mathbb{CP}^n$ to give it the structure of a complex manifold.

Definition 1.9 (Algebraic variety, rough definition). An algebraic variety is a topological space X with a finite covering by $U_1, \dots, U_m$ open subsets such that $U_i \cong Z(I_i) \subset \mathbb{C}^{j_i}$

Remark 1.14. We can show that two different closed subvarieties $Z(I) \subseteq \mathbb{C}^n$ and $Z(J) \subseteq \mathbb{C}^n$ are isomorphic as algebraic varieties if and only if the corresponding finitely generated coordinate rings (generated algebras) $\mathbb{C}[x_1, \dots, x_n]/I$ and $\mathbb{C}[x_1, \dots, x_n]/J$ are isomorphic. It is a natural statement, since points are in bijections with maximal ideals in these rings. We think of them as just the same variety in two different realizations. Such varieties are called *affine*.

Remark 1.15. The main advantage of $\mathbb{CP}^n$ over $\mathbb{C}^n$ is that it is compact. Intuitively, it is the “smallest” way to compactify $\mathbb{C}^n$ to a compact complex manifold. In contrast, if we just wanted a compactification as a real manifold, we could do it with adding a single point.

Theorem 1.6. $\mathbb{CP}^n$ is a compact complex manifold.

Proof. We can take the quotient in two steps. First by $\mathbb{R}_{>0}$, then by $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$. We immediately see that (as real manifolds):

$$(\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{R}_{>0} \cong \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid |x_0|^2 + \dots + |x_n|^2 = 1\} \cong S^{2n+1}.$$

Then $\mathbb{CP}^n$ is the image of the compact set S^{2n+1} under a continuous map and is therefore compact. $\square$

Definition 1.10. Consider $\mathbb{C}[x_0, \dots, x_n]$. Notice that this is a graded ring $\mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$, where:

$$S_d := \text{span} \left\{ x_0^{a_0} \cdots x_n^{a_n} \mid \sum a_i = d \right\},$$

which are the homogeneous polynomials of degree d .

Definition 1.11. An ideal $I \triangleleft \mathbb{C}[x_0, \dots, x_n]$ is called homogeneous if it is generated by homogeneous polynomials (recall $\mathbb{C}[x_0, \dots, x_n]$ is Noetherian and thus generated by finitely many polynomials).

For a homogeneous ideal $J \triangleleft \mathbb{C}[x_0, \dots, x_n]$ we define:

$$Z_+(J) := \{(a_0 : \dots : a_n) \in \mathbb{CP}^n \mid f(a_0, \dots, a_n) = 0 \quad \forall \text{ homogeneous } f \in J\},$$

which is well-defined since f indeed vanishes on all of $(a_0 : \dots : a_n)$ since it is homogeneous. These are called projective algebraic subsets.

Similarly, we define for $V \subset \mathbb{CP}^n$ the set:

$$I_+(V) := \{\text{homogeneous } f \in \mathbb{C}[x_0, \dots, x_n] \mid f(p) = 0 \quad \forall p \in V\},$$

which is a homogeneous ideal since it is generated by homogeneous ideals.

Remark 1.16. The Zariski topology on $\mathbb{CP}^n$ is defined by declaring the closed subsets to be the $Z_+(J)$.

Remark 1.17. We saw that $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ where $U_i \cong \mathbb{C}^n$.

Question. Is the topology induced from the Zariski topology on $\mathbb{CP}^n$ the same Zariski topology we defined on $\mathbb{C}^n$?

Answer. Yes, because $U_i \cap Z_+(J) = Z(\tilde{T}^i)$ where

$$\tilde{T}^i = \langle f(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \in \mathbb{C}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \mid \forall f \in J \text{ homogeneous}) \rangle$$

This process is called dehomogenization.

Question. What is the relation between $Z_+(J) \subset \mathbb{CP}^n$ and $Z(J) \subset \mathbb{C}^{n+1}$? The set $Z(J)$ is called the affine cone over $Z_+(J)$.

Remark 1.18. Relationship between closed subsets of $\mathbb{CP}^n$ and ideals is a little more complicated.

Remark 1.19. If J is homogeneous then $J = \bigoplus_{d \geq 0} J_d$ and $Z_+(J) = Z_+(J')$ if there exists d_0 such that $J_d = J'_d$ for all $d \geq d_0$. (throwing out the first few terms).

Example 1.8 (Irrelevant ideal). The ideal $(x_0, x_1, \dots, x_n)$ is called the irrelevant ideal. This is the unique homogeneous maximal ideal.

Example 1.9. Consider $\{x_0x_2 - x_1^2 = 0\} \subset \mathbb{CP}^2$. If we dehomogenize with respect to x_0 we set it to 1 and get:

$$\bigcap U_0: \{y_2 - y_1^2\} \subset \mathbb{C}^2.$$

Example 1.10. Smooth quadric degree 2 homogeneous polynomial $\{x_0x_3 - x_1x_2 = 0\} \subset \mathbb{CP}^3$. We will show that this is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Example 1.11. $\{y^2 - 4x^3 - g_2xz^2 - g_3z^3 = 0\} \subset \mathbb{CP}^2$ with coordinates $(x : y : z)$. This is known as an elliptic curve.

2 Bezout's theorem

Remark 2.1 (Manifestation of compactness). Let I be a homogeneous ideal. Then we can define:

$$a_d := \dim_{\mathbb{C}} (\mathbb{C}[x_0, \dots, x_n]/I)_{\deg=d} < \infty.$$

Theorem 2.1. For a nontrivial (not irrelevant for example) homogeneous ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$, the coefficients a_d satisfy:

- (1) There exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(d) = a_d$ for all sufficiently large $d > 0$. The polynomial P is called the Hilbert polynomial of $Z_+(I)$.
- (2) $\deg(P) = \dim Z_+(I)$.
- (3) $(\dim Z_+(I)) \mid (\text{leading coefficient of } P) \in \mathbb{N}$, which is called the degree of $Z_+(I)$.
- (4) The Hilbert function $f(t) := \sum_{d \geq 0} a_d t^d$ is a rational function of the form

$$f(t) = \frac{g(t)}{(1-t)^{\dim Z(I)+1}},$$

with a polynomial $g(t)$ such that $g(1) \neq 0$.

Example 2.1. Take the homogeneous ideal $I = \{0\}$ (which corresponds to the whole space). We have that a_d is the dimension of the space of degree d of homogeneous polynomials in $n+1$ variables. This is equal to the number of ways of picking n separators among $n+d$ dots. Thus, we get for all $d \geq 0$

$$a_d := \binom{n+d}{n} = \frac{1}{n!} (d+n) \cdots (d+1) = \frac{1}{n!} d^n + \text{etc.},$$

times dimension factorial the leading coefficient is 1, and this is indeed the dimension which we expect.

The generating function:

$$\sum_{d \geq 0} \frac{t^d}{n!} (d+n) \cdots (d+1) = \sum_{d \geq -n} \frac{t^d}{n!} (d+n) \cdots (d+1) = \frac{1}{n!} \left(\frac{\partial}{\partial t} \right)^n \left(\frac{1}{1-t} \right) = \frac{1}{(1-t)^{n+1}},$$

so the claims of the previous theorem hold.

Example 2.2. Consider a curve $C \subset \mathbb{CP}^2$ given by $I = (f)$, for a homogeneous $f \in \mathbb{C}[x_0, x_1, x_2]$ of degree d without repeated factors.

We have that:

$$\dim(\mathbb{C}[x_0, x_1, x_2]/I)_{\deg=d} = \dim(\mathbb{C}[x_0, x_1, x_2])_{\deg=d} - \dim(\mathbb{C}[x_0, x_1, x_2])_{\deg=d-k}$$

because of $xf: \mathbb{C}[x_0, x_1, x_2] \rightarrow \mathbb{C}[x_0, x_1, x_2]$ is injective. Thus,

$$a_d = \begin{cases} \frac{(d+2)(d+1)}{2} - \frac{(d-k+2)(d-k+1)}{2}, & d \geq k; \\ \frac{(d+2)(d+1)}{2}, & 0 \leq d < k. \end{cases}$$

Thus, for sufficiently large d we get

$$a_d = dk - \frac{k(k-3)}{2},$$

which is linear in d . We get that $\deg Z_+(I) = k$. And the generating function is:

$$f(t) := \frac{1}{(1-t)^3} - \frac{t^k}{(1-t)^3} = \frac{\sum_{i=0}^{k-1} t^i}{(1-t)^2},$$

so again the last theorem holds.

Definition 2.1. A $\bigoplus S_d = \mathbb{C}[x_0, \dots, x_n]$ -module M is called a graded module if $M = \bigoplus_{d \in \mathbb{Z}} M_d$, where M_d is an abelian group such that $S_e \cdot M_d \subset M_{d+e}$.

Definition 2.2 (Shifted module). The shifted module $M(r)$ has the same module structure but with shifted grading:

$$M(r)_{\deg=k} := M_{\deg=k+r}.$$

For any graded module M we get a (degree 0) graded map of graded modules $M(-1) \xrightarrow{x_n} M$ (multiplication by x_n ?) with kernel K and cokernel C (also graded modules). This implies that:

$$0 \longrightarrow K \longrightarrow M(-1) \longrightarrow M \longrightarrow C \longrightarrow 0$$

is an exact sequence of graded modules. Thus,

$$0 \longrightarrow K_d \longrightarrow M(-1)_d \longrightarrow M_d \longrightarrow C_d \longrightarrow 0$$

is an exact sequence of vector spaces for all $d \geq 0$, and thus

$$-\dim K_d + \dim M(-1)_d + \dim M_d + \dim C_d = 0.$$

This happens if and only if

$$-f_k(t) + f_{M(-1)}(t) - f_M(t) + f_C(t) = 0,$$

then we observe that $f_{M(-1)}(t) = t f_M(t)$ we get:

$$(1-t)f_M(t) = f_C(t) - f_K(t).$$

Upshot: $\mathbb{C}[x_0, \dots, x_n]$ -module section on K and C factors through $\mathbb{C}[x_0, \dots, x_{n-1}]$ -module section $\mathbb{C}[x_0, \dots, x_n]/(x_n)$ because x_n goes to 0 on K and C . We now apply induction to get (4). The proof of (1)-(3) is significantly more difficult and is thus omitted.

Remark 2.2. We defined $\dim Z(I)$ for irreducible $Z(I)$

By analogy we get for irreducible $Z_+(I)$ that:

$$\dim Z_+(I) \cap U_0 = \text{tr. deg. Frac}(\mathbb{C}[x_1, \dots, x_n]/\tilde{I}).$$

Maybe on any other U_i , say U_1 , we get a different definition? NO! Because

Remark 2.3. Expressing $\text{Frac}(A) := \text{Frac}(\mathbb{C}[x_1, \dots, x_n]/\tilde{I})$ intrinsically in terms of the variables $x_1, \dots, x_n$ we see that:

$$\text{Rat}(\mathbb{Z}_+(I)) = \left\{ \frac{g_1}{g_2}, \deg g_1 = \deg g_2, g_1, g_2 \in A \right\} \subset \text{Frac}(A).$$

Theorem 2.2 (Bezout's theorem). *Let $C_1, C_2 \subset \mathbb{CP}^2$ be curves with no common components. Then the number of points in $C_1 \cap C_2$ counting multiplicities is $(\deg C_1)(\deg C_2)$.*

Remark 2.4. Suppose that C_1 and C_2 intersect at $(1 : 0 : 0) \in \mathbb{CP}^2$ and $(1 : 0 : 0) \in U_0 \cong \mathbb{C}^2 \subset \mathbb{CP}^2$. Let $C_1 \cap U_0 = Z(f)$, $C_2 \cap U_0 = Z(g)$ for some $f, g \in \mathbb{C}[y_1, y_2]$ for $y_i = \frac{x_i}{x_0}$.

Let

$$R = \left\{ \frac{F(y_1, y_2)}{G(y_1, y_2)} \mid F, G \in \mathbb{C}[y_1, y_2] \text{ s.t. } G(0, 0) \neq 0 \right\}.$$

Because f and g have no irreducible components in common (since C_1 and C_2 have no common components). This implies that $\dim_G R/(f, g) < \infty$. We also define:

$$\text{mult}_{(1:0:0)}(\{f=0\} \cap \{g=0\}) := \dim_G R/(f, g) < \infty$$

Example 2.3. Let $C_1 = Z_+(x_1)$ and $C_2 = Z_+(x_2)$. In U_0 we get $f = y_1$, $g = y_2$, and thus $R/(y_1, y_2) = \mathbb{C}$, and so $\dim R/(y_1, y_2) = 1$, which is exactly the number of intersection the lines C_1 and C_2 should have according to Bezout's theorem.

Example 2.4. Let $C_1 = Z_+(x_2)$ and $C_2 = Z_+(x_0x_2 - x_1^2)$. In U_0 we get $f = y_2$, $g = y_2 - y_1^2$, and then:

$$\begin{aligned} R/(f, g) &= (R/(f))/(g) = \left\{ \frac{F(y_1)}{G(y_1)} \mid G(0, 0) \neq 0 \right\} / (y_2 - y_1^2) \\ &= \left\{ \frac{F(y_1)}{G(y_1)} \mid G(0, 0) \neq 0 \right\} / (y_1^2) = \mathbb{C} \cdot 1 + \mathbb{C} \cdot y_1 \cong \mathbb{C}^2, \end{aligned}$$

so at $(1 : 0 : 0)$ the multiplicity of $C_1 \cap C_2$ is 2.

2.1 Proof of Bezout's theorem

The idea of the proof is to consider $M = \mathbb{C}[x_0, x_1, x_2]/(f_1, f_2)$ as a graded $\mathbb{C}[x_0, x_1, x_2]$ -module.

Proposition 2.3. *For every nonzero finitely generated graded module M over a graded Noetherian ring A there exists a finite filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_k = M$$

so that for each $i \in \{1, \dots, k\}$ the quotient module M_i/M_{i-1} is isomorphic to $(A/p_i)(d_i)$, where $p_i \subset A$ is a homogeneous prime ideal in A and $d_i \in \mathbb{Z}$ is a grading shift.

Proof. When $M = \mathbb{C}[x_0, x_1, x_2]/(f_1, f_2)$, from the description of M_i/M_{i-1} as $(A/p_i)(d_i)$ we see $p_i = \text{Ann}(m) := \{a \in A \text{ s.t. } am = 0\}$. So any such $p \supseteq (f_1, f_2)$ if and only if $Z_+(p) \subset Z_+(f_1, f_2)$. $\square$

Remark 2.5. The Hilbert polynomial of any finitely generated graded $\mathbb{C}[x_0, x_1, x_2]$ -module is $\sum_{i=1}^h \text{HP}((A/p_i)(d_i))$ (the Hilbert polynomial of the summands).

Note that $\text{HP}(\mathbb{C}[x_0, x_1, x_2]/(f_1, f_2))$ can be computed since: Note that the map labeled f_2 is injective since otherwise f_1 and f_2 have a common factor.

We also label the terms in the sequence by $M_1(-\deg f_2)$, M_1 and M respectively.

Note that as we have computed before $P_{M_1}(k) = (\deg f_1)k + c$ and $P_M(k) = (h \deg f_1 + c) - ((k - \deg f_2) \deg f_1 + 1) = \deg f_1 \deg f_2 = \deg C_1 \deg C_2$.

Next consider the irrelevant ideal $P = (x_0, x_1, x_2)$. We get that $\mathbb{C}[x_0, x_1, x_2]/P \cong \mathbb{C}$ so for $d = 0, 1, 2, \dots$ we get $a_d = 1, 0, 0, \dots$, so $P_{\mathbb{Q}[x_0, x_1, x_2]/P}(h) = 0$.

Next consider $P = (x_1, x_2)$ (the other cases are symmetrical). We have the isomorphism $\mathbb{C}[x_0, x_1, x_2]/P \cong \mathbb{C}[x_0]$, and thus for $d = 0, 1, 2, \dots$ we get $a_d = 1, 1, 1, \dots$ and therefore $P_{\mathbb{Q}[x_0, x_1, x_2]/P}(k) = 1$.

Let $S \subset A$ (where A is a ring) be a mult subsets i.e. $1 \in S$ and $f, g \in S$ implies $fg \in S$.

The localization of A at S , A_s is the set $\left\{ \frac{a}{s} \mid a \in A \text{ and } s \in S \right\} / \sim$ where

$$\frac{a_1}{s_1} = \frac{a_2}{s_2}$$

if and only if there exists $s_3 \in S$ such that $s_3(s_1a_2 - s_2a_1) = 0$ in A .

Consider the map $\varphi: A \rightarrow A_s$ given by $a \mapsto \frac{a}{1}$. For an A -module

$$M_S := \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \sim$$

where

$$\frac{m_1}{s_1} = \frac{m_2}{s_2}$$

if and only if there exists $s_3 \in S$ such that $s_3(s_1m_2 - s_2m_1) = 0$ in M .

Define $\varphi_{M,S}: M \rightarrow M_s$ defined by $m \mapsto \frac{m}{1}$.

Remark 2.6 (Fact). Localizing at S is an exact functor. i.e. if

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence (SES) of A -modules, then so is the corresponding short exact sequence of A_S -modules!

$$0 \longrightarrow M_S \longrightarrow M'_S \longrightarrow M''_S \longrightarrow 0$$

Remark 2.7. Suppose our $P = (x_1, x_2)$. Consider:

$$S = \{ \text{homogeneous polynomials of } \mathbb{C}[x_0, x_1, x_2] \text{ that don't vanish at } (1 : 0 : 0) \}$$

Notice that $x_0 \in S$ so we can localize at x_0 first (i.e. take $T = \{x_0^n \mid n \geq 0\}$) and only then we divide $\mathbb{C}[x_0, x_1, x_2]_{x_0} := \mathbb{C}[x_0, x_1, x_2]/P$. Then we get $\mathbb{C}[\frac{x_1}{x_0}, \frac{x_2}{x_0}, x_0^{\pm 1}] = \mathbb{C}[y_1, y_2][x_0^{\pm 1}]$. Localizing all the way we get that $\mathbb{Z}$ -graded ring $R[x_0^{\pm 1}]$ where $R = \left\{ \frac{F(y_1, y_2)}{G(y_1, y_2)} \mid G(0, 0) \neq 0 \right\}$.

Consider $M_S \cong \left(R/(\tilde{f}_1, \tilde{f}_2) \right) [x_0^{\pm 1}]$ where $\tilde{f}_i = \frac{f_i}{x_0^{\deg f_i}}$ the dehomogenization of the f_i , the degree 0 part is $R/(f, g)$ from the definition of

Corollary 2.4. Let $C \subset \mathbb{CP}^2$ be an irreducible $\deg C = d$ curve. Then C admits a degree d morphism onto $\mathbb{CP}^1$.

Proof. Suppose $P := (1 : 0 : 0) \notin C$. Then the stereographic projection $\varphi: \mathbb{CP}^2 - \{P\} \rightarrow \mathbb{CP}^1$ from P given by $(x_0 : x_1 : x_2) \mapsto (x_1 : x_2)$. $\square$

Definition 2.3 (Gonality). The gonality of a curve C is defined to be:

$$\text{gon}(C) := \min \left\{ \deg f \mid f: C \rightarrow \mathbb{RP}^1 \text{ is nonconstant} \right\}.$$

3 Line bundles and vector bundles

Remark 3.1. If $f: X \rightarrow \mathbb{C}^N$ is a holomorphic function on a compact complex manifold, then f is constant. In particular this is true for $f: X \rightarrow \mathbb{C}$.

Definition 3.1 (Vector bundle). A vector bundle of rank r over X is a complex manifold W with a holomorphic map $\pi: W \rightarrow X$ such that for every $p \in X$ there exists an open set $U \ni p$ and a biholomorphism $\varphi_U: U \times \mathbb{C}^r \rightarrow \pi^{-1}(U)$ so that the diagram

$$\begin{array}{ccc} U \times \mathbb{C}^r & \xrightarrow{\quad} & \pi^{-1}(U) \\ & \searrow & \swarrow \\ & U & \end{array}$$

commutes and is compatible with the complex vector space structures on the fibers.

Remark 3.2. A vector bundle of rank 1 is called a line bundle.

Example 3.1 (Real vector bundle). The Möbius strip. The trivial bundle and the infinite mobius strip over S^1 .

Definition 3.2 (Trivial vector bundle). A vector bundle W of rank r is called a trivial vector bundle if $W \cong X \times \mathbb{C}^r$ as a vector bundle. That is, there exists a biholomorphic map $\psi: W \rightarrow X \times \mathbb{C}^r$ such that

$$\begin{array}{ccc} W & \xrightarrow{\quad} & X \times \mathbb{C}^r \\ & \searrow \pi & \swarrow pr_1 \\ & U & \end{array}$$

Remark 3.3. Everything you can do to vector spaces in linear algebra you can do to vector bundles.

Definition 3.3 (Picard group).